

REGULARITY OF THE OPTIMAL STOPPING PROBLEM FOR LÉVY PROCESSES WITH NON-DEGENERATE DIFFUSIONS

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ABSTRACT. The value function of an optimal stopping problem for a process with Lévy jumps is known to be a generalized solution of a variational inequality. Assuming the diffusion component of the process is nondegenerate and a mild assumption on the singularity of the Lévy measure, this paper shows that the value function of obstacle problems on an unbounded domain with infinite activity jumps is $W_{p,loc}^{2,1}$.

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1. INTRODUCTION

This paper analyzes the finite horizon optimal stopping problem for an n -dimensional jump diffusion process X which is governed by the following stochastic differential equation:

$$(1.1) \quad dX_t = b(X_{t-}, t) dt + \sigma(X_{t-}, t) dW_t + d\mathcal{J}_t,$$

in which $W = \{W_t; t \geq 0\}$ is the d -dimensional standard Brownian motion and $\mathcal{J} = \{\mathcal{J}_t; t \geq 0\}$ is a pure jump Lévy process independent of the Brownian motion. This jump process \mathcal{J} can be of finite/infinite activity with

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finite/infinite variation. We denote the Lévy measure of \mathcal{J} as ν (please refer to Section 2 for the definition of \mathcal{J} and its properties).

We investigate the problem of maximizing the discounted terminal reward g by optimally stopping the process X before a fixed time horizon T . The value function of this problem is defined as

$$(1.2) \quad u(x, t) = \sup_{\tau \in \mathcal{T}_{0, T-t}} \mathbb{E}^{t, x} [e^{-r\tau} g(X_\tau)],$$

in which $\mathcal{T}_{0, t}$ is the set of all stopping times valued between 0 and t . A specific example of such an optimal stopping problem is the American option pricing problem, where X models the logarithm of the stock price processes and g represents the pay-off function. In [1], Ait-Sahalia and Jacod consider the model in (1.1) and find evidence of infinitely active jumps in stock prices.

The function u satisfies, a variational inequality with a nonlocal integral term (see e.g. Chapter 3 of [6]). In the literature different solution concepts were studied. Pham showed in [27] that the value function of the optimal stopping problem for a controlled jump process is a viscosity solution of a variational inequality using the dynamic programming principle. Also see [2], [3] for more recent results in this direction. In [23], Lamberton and Mikou proved that the value function associated to the optimal stopping problem for Lévy processes can be understood as the unique solution of the same variational inequality in the distributional sense. In general, without assuming that the diffusion coefficient is non-degenerate, the value function is not expected to be a smooth solution of this variational inequality.

Using the non-degeneracy assumption, Sections 1-3 in Chapter 3 of [6] and in [16] developed the regularity results for the Cauchy problem and boundary value problems for second order partial integro-differential equations. They proved existence and uniqueness of solutions in both Sobolev and Hölder spaces. Also see [24]. On the other hand, there are only limited results for obstacle problems either finite or infinite activity jumps. Bensoussan and Lions showed in Theorem 4.4 pp.250 of [6] that the solution of a variational inequality on a bounded domain can be characterized as an element in a certain Sobolev space. Their regularity results are not enough to ensure the smooth-fit property. Later, these results were extended to problems on unbounded domains by [19] and [32], but these papers assumes that the state variable X follows a diffusions or a jump diffusions with finite activity jumps. [26], [31], and [4] further analyzed the variational inequality for jump diffusions with finite activity jumps using different techniques. More recently, [11], generalizing the results in [18] for the diffusion case, analyzed the regularity of the value function of infinite horizon impulse control problems for infinite activity but finite variation jumps. A regularity result which treats the obstacle problem on unbounded domains with infinite variation jumps has been missing in the literature.

In this paper we extend the results of [6] and analyze the optimal stopping problem for Lévy processes with infinite activity jumps on an unbounded domain. In our main result (Theorem 4.1), we show that the value function, which is the unique solution of a variational inequality, is an element of $W_{p, loc}^{2,1}$ (see Section 4.1 for the definition of this Sobolev space). This regularity result directly implies that the smooth fit property holds and the value function is smooth inside the continuation region.

We use the penalty method to analyze the variational inequality. The infinite activity unbounded jumps bring technical difficulties in applying this method. In particular, the usual L_p -estimates are no longer directly applicable. Our main technical contribution is the norm estimate in Theorem 5.1, which we prove using an interior estimation technique.

In Section 3, we treat the finite variation jump case separately because the reduced form of the integral operator and the Lipschitz continuity of the value function enable us to use classical arguments to prove that the value function is $C^{2,1}$ the continuation region. We present this section because contrasting it with the later sections

clearly demonstrates the technical difficulties of handling the infinite variation case due to the complicated form of the integral operator in this case.

The rest of the paper is organized as follows. In Section 2, we introduce the variational inequality and recall two notions of generalized solutions studied in [27] and [23]. In Section 3, we discuss the finite variation jump case and analyze the regularity of value function in the continuation region. Section 4 is devoted to study the regularity when jumps may have infinite variation. The main result of this paper (Theorem 4.1) is proved in Section 5. A key estimate (Theorem 5.1) is proved in Section 6. Proofs of several auxiliary lemmas are given in the Appendix.

2. PRELIMINARIES

2.1. A priori regularity of the value function. Let us first analyze the pure jump component \mathcal{J} in (1.1). According to the Lévy-Itô decomposition (see e.g. Theorem 19.2 in [28]), \mathcal{J} can be decomposed as

$$(2.1) \quad \mathcal{J}_t = \mathcal{J}_t^\ell + \lim_{\epsilon \downarrow 0} \mathcal{J}_t^\epsilon,$$

in which

$$(2.2) \quad \mathcal{J}_t^\ell = \int_0^t \int_{|y|>1} y \mu(ds, dy), \quad \mathcal{J}_t^\epsilon = \int_0^t \int_{\epsilon \leq |y| \leq 1} y \tilde{\mu}(ds, dy),$$

represent large and small jumps respectively. Here μ is a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \{0\})$. Its mean measure is the Lévy measure ν , which is a positive Radon measure on $\mathbb{R}^n \setminus \{0\}$ with a possible singularity at 0. Even with this possible singularity at 0, the measure ν still satisfies

$$(2.3) \quad \int_{\mathbb{R}^n} (|y|^2 \wedge 1) \nu(dy) < +\infty.$$

Here, the norm $|\cdot|$ is the standard Euclidean norm: $|y| \triangleq (\sum_{i=1}^n (y^i)^2)^{1/2}$. In (2.2), $\tilde{\mu}(ds, dy) = \mu(ds, dy) - ds \nu(dy)$ is the compensated Poisson measure.

We assume that the drift and the volatility in (1.1) are bounded and Lipschitz continuous, i.e., there exists a positive constant $L_{b,\sigma}$ such that

$$(H1) \quad \begin{aligned} &|b(x, t) - b(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq L_{b,\sigma} |x - y|, \quad \forall x, y \in \mathbb{R}^n, \\ &\text{moreover, } |b(x, t)| \text{ and } |\sigma(x, t)| \text{ are bounded on } \mathbb{R}^n \times [0, T]. \end{aligned}$$

We name the solution of (1.1), with the initial condition $X_0 = x$, as X^x . Thanks to (H1), X^x has the following norm estimates.

Lemma 2.1. *Let us assume b and σ satisfy (H1). Then there exists a positive constant C such that*

$$(2.4) \quad \mathbb{E} |X_\tau^x - X_\tau^y| \leq C |x - y|, \quad \text{for any } \tau \in \mathcal{T}_{0,t} \text{ with } t \leq T \text{ and } x, y \in \mathbb{R}^n.$$

Moreover, if the Lévy measure satisfies

$$(H2) \quad \int_{|y|>1} |y| \nu(dy) < +\infty,$$

then we have

$$(2.5) \quad \mathbb{E} |X_\tau^x| \leq C, \quad \mathbb{E} |X_\tau^x - x| \leq C t^{1/2}, \quad \text{and} \quad \mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s^x - x| \right] \leq C t^{1/2}.$$

Remark 2.1. *Similar estimates were given in Lemma 3.1 of [27] under a slightly stronger assumption on the large jumps: $\int_{|y|>1} |y|^2 \nu(dy) < +\infty$. Using the equivalence between the norm $|y|$ and the norm $\sum_{i=1}^n |y^i|$, one could prove Lemma 2.1 under assumption (H2). We give its proof in Appendix A.*

Let us assume that the terminal reward $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a bounded and Lipschitz continuous function, i.e., there exist positive constants K and L such that

$$(H3) \quad 0 \leq g(x) \leq K \quad \text{and}$$

$$(H4) \quad |g(x) - g(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Thanks to (H3), u is uniformly bounded by K . Moreover, (H4) and norm estimates of X in Lemma 2.1 ensure that u has the regularity properties given in the next Lemma. The proof is omitted since it is the same as the proof of Proposition 3.3 in [27], once we replace Lemma 3.1 of [27] by our Lemma 2.1.

Lemma 2.2. *Let us assume that g satisfies (H3) and (H4). Then there exists a constant $L_x > 0$ such that*

$$(2.6) \quad |u(x_1, t) - u(x_2, t)| \leq L_x |x_1 - x_2|, \quad \text{for any } x_1, x_2 \in \mathbb{R} \text{ and } t \in [0, T].$$

Moreover, if the Lévy measure satisfies (H2), then there exists a constant $L_t > 0$ such that

$$(2.7) \quad |u(x, t_1) - u(x, t_2)| \leq L_t |t_1 - t_2|^{1/2}, \quad \text{for any } t_1, t_2 \in [0, T] \text{ and } x \in \mathbb{R}.$$

The Lipschitz continuity of $u(\cdot, t)$ and the semi-Hölder continuity of $u(x, \cdot)$ will be useful to show further regularity properties of u in the next three sections.

For the optimal stopping problem, we define the continuation region \mathcal{C} and the stopping region \mathcal{D} as usual:

$$\mathcal{C} \triangleq \{(x, t) \in \mathbb{R}^n \times [0, T] : u(x, t) > g(x)\} \quad \text{and} \quad \mathcal{D} \triangleq \{(x, t) \in \mathbb{R}^n \times [0, T] : u(x, t) = g(x)\}.$$

2.2. The variational inequality. Intuitively, one can expect from the Itô's Lemma for Lévy processes (see e.g. Proposition 8.18 in [9] pp. 279) that the value function u , defined in (1.2), satisfies the following variational inequality:

$$(2.8) \quad \begin{aligned} \min \{(-\partial_t - \mathcal{L} + r)u(x, t), u(x, t) - g(x)\} &= 0, \quad (x, t) \in \mathbb{R}^n \times [0, T], \\ u(x, T) &= g(x), \end{aligned}$$

in which the integro-differential operator \mathcal{L} , the infinitesimal generator of X , is defined via a bounded test function ϕ as

$$(2.9) \quad \mathcal{L}\phi(x, t) \triangleq \mathcal{L}_D\phi(x, t) + I\phi(x, t), \quad \text{with} \quad \mathcal{L}_D\phi(x, t) \triangleq \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \sum_{i=1}^n b_i(x, t) \frac{\partial \phi}{\partial x^i}.$$

Here $A = (a_{ij})_{n \times n} \triangleq \frac{1}{2} \sigma(x, t) \sigma(x, t)^T$ is a $n \times n$ matrix and the integral term

$$(2.10) \quad \begin{aligned} I\phi(x, t) &\triangleq \int_{\mathbb{R}^n} \left[\phi(x + y, t) - \phi(x, t) - \sum_{i=1}^n y^i \frac{\partial \phi}{\partial x^i}(x, t) 1_{\{|y| \leq 1\}} \right] \nu(dy) \\ &= \int_{\mathbb{R}^n} [\phi(x + y, t) - \phi(x, t) - y \cdot \nabla_x \phi(x, t) 1_{\{|y| \leq 1\}}] \nu(dy). \end{aligned}$$

However, one does not know a priori whether the value function u is sufficiently regular (i.e., $u \in C^{2,1}(\mathbb{R}^n \times [0, T])$) to justify applying Itô's Lemma. Moreover, the integral term $I\phi(x, t)$ is only well defined in classical sense when ϕ has certain regularity properties. It is sufficient to require ϕ to be a bounded function in $C^1(B_\epsilon(x))$, in which $B_\epsilon(x)$ is an open ball in \mathbb{R}^n centered at x with some radius $\epsilon \in (0, 1)$, and that $\nabla_x \phi(\cdot, t)$ to be Lipschitz in $B_\epsilon(x)$ uniformly in t , i.e., for $t \in [0, T]$ there exists a positive constant L_B such that $|\nabla_x \phi(x_1, t) - \nabla_x \phi(x_2, t)| \leq L_B |x_1 - x_2|$, for $x_1, x_2 \in B_\epsilon(x)$. Indeed, using these regularity properties of ϕ we have that

$$(2.11) \quad I\phi(x, t) = I_\epsilon \phi(x, t) + I^\epsilon \phi(x, t) < \infty, \quad \text{where}$$

$$(2.12) \quad I^\epsilon \phi(x, t) = \int_{|y| > \epsilon} [\phi(x + y, t) - \phi(x, t)] \nu(dy) - \nabla_x \phi(x, t) \cdot \int_{\epsilon < |y| \leq 1} y \nu(dy),$$

$$(2.13) \quad \begin{aligned} I_\epsilon \phi(x, t) &= \int_{|y| \leq \epsilon} [\phi(x + y, t) - \phi(x, t) - y \cdot \nabla_x \phi(x, t)] \nu(dy) \\ &= \int_{|y| \leq \epsilon} \sum_{i=1}^n y^i (\partial_{x^i} \phi(z_i, t) - \partial_{x^i} \phi(x, t)) \nu(dy) \leq \int_{|y| \leq \epsilon} L_B |y|^2 \nu(dy). \end{aligned}$$

(In (2.13), the second equality follows from the mean value theorem with some $z_i \in \mathbb{R}^n$ satisfying $|z_i - x| < |y|$, while the inequality follows from the Lipschitz continuity of $\nabla_x \phi(\cdot, t)$. Note that $\int_{\epsilon < |y| \leq 1} |y| \nu(dy) \leq \frac{1}{\epsilon} \int_{\epsilon < |y| \leq 1} |y|^2 \nu(dy) < +\infty$, we obtain the inequality in (2.11).)

Given the regularity of u in Lemma 2.2, it is not clear that u has the Lipschitz continuous first derivative to ensure that Iu is well defined in the classical sense. This will be addressed in Section 4.

2.3. Viscosity solutions. We will introduce the viscosity solution concept following [27]. Let us define

$$C_1(\mathbb{R}^n \times [0, T]) \triangleq \left\{ \phi \in C^0(\mathbb{R}^n \times [0, T]) : \sup_{(x, t) \in \mathbb{R}^n \times [0, T]} \frac{|\phi(x, t)|}{1 + |x|} < +\infty \right\}.$$

We adapt the notion of viscosity solutions used in Definition 2.1 of [27] into our context and give the following definition. (We assume that (H2) holds so that $I\phi(x, t)$ is well defined for $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap C_1(\mathbb{R}^n \times [0, T])$. Indeed, for $\phi \in C_1(\mathbb{R}^n \times [0, T])$, we have $|\phi(x + y, t) - \phi(x, t)| \leq C(1 + |y|)$ for some C independent of y . Therefore, $\int_{|y| > \epsilon} [\phi(x + y, t) - \phi(x, t)] \nu(dy) < +\infty$ in (2.12) thanks to (H2).)

Definition 2.1. (i) Any $u \in C^0(\mathbb{R}^n \times [0, T])$ is a viscosity supersolution (subsolution) of (2.8) if

$$(2.14) \quad \min \{ -\partial_t \phi - \mathcal{L} \phi + ru, u(x, t) - g(x) \} \geq 0 \ (\leq 0),$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap C_1(\mathbb{R}^n \times [0, T])$ such that $u(x, t) = \phi(x, t)$ and $u(\tilde{x}, \tilde{t}) \geq \phi(\tilde{x}, \tilde{t})$ ($u(\tilde{x}, \tilde{t}) \leq \phi(\tilde{x}, \tilde{t})$) for all $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [0, T]$.

(ii) u is a viscosity solution of (2.8) if it is both supersolution and subsolution.

As in [27], we have the following result.

Proposition 2.1. If the Lévy measure ν satisfies (H2), the value function $u(x, t)$ is the unique viscosity solution of (2.8).

2.4. Standing assumptions on the Lévy Measure. In addition to (H2), throughout this paper, we assume that the Lévy measure ν has a density, which we denote by $\rho(y)$, and there exists a positive constant M such that

$$(H5) \quad \rho(y) \leq \frac{M}{|y|^{n+\alpha}}, \quad \text{for } |y| \leq 1 \text{ and some constant } \alpha \in [0, 2).$$

Remark 2.2. The Lévy measures ν , corresponding to Lévy processes widely used in the financial modelling for the single asset case, satisfy (H5) with $n = 1$.

In jump diffusions models where ν is a probability measure, if the density $\rho(y)$ is bounded, (H5) is satisfied with sufficiently large M . Examples of this case are Merton's model and Kou's model. On the other hand, if $\rho(y) \in C^0(B_1(0) \setminus \{0\})$ and $\rho(y)$ has a power singularity $1/|y|^\beta$ with $0 < \beta < 1$ at $y = 0$, (H5) is again fulfilled because $\frac{1}{|y|^{1+\alpha}} > \frac{1}{|y|^\beta}$ for any $\alpha \geq 0$ and $|y| \leq 1$.

Moreover, for Lévy processes that are the Brownian motion subordinated by tempered stable subordinators, it follows from (4.25) in [9] that ρ has a power singularity $1/|y|^{1+2\beta}$, with $0 \leq \beta < 1$, at $y = 0$. Therefore (H5) is

satisfied by choosing $\alpha = 2\beta$ and sufficiently large M . In particular, this class of Lévy processes contains Variance Gamma and Normal Inverse Gaussian where $\beta = 0$ or $1/2$ respectively.

Furthermore, for the generalized tempered stable processes (see Remark 4.1 in [9]) whose Lévy density is $\rho(y) = \frac{C_-}{|y|^{1+\alpha_-}} e^{-\lambda_-|y|} 1_{\{y<0\}} + \frac{C_+}{|y|^{1+\alpha_+}} e^{-\lambda_+|y|} 1_{\{y>0\}}$, with $\alpha_-, \alpha_+ < 2$, (H5) is satisfied by choosing $\alpha = \max\{\alpha_-, \alpha_+, 0\}$ and $M = \max\{C_-, C_+\}$. In particular, CGMY processes in [8] are special examples of generalized tempered stable processes. In the similar manner, one can also check that the regular Lévy processes of exponential type (RLPE) in [7] also satisfy (H5).

3. FINITE VARIATION JUMPS AND REGULARITY IN THE CONTINUATION REGION

In this section, we will analyze the regularity of u in the continuation region, when jumps of X have finite variation, i.e.,

$$(3.1) \quad \int_{\mathbb{R}^n} (|y| \wedge 1) \nu(dy) < +\infty.$$

It is clear that (3.1) holds when we assume (H5) is satisfied with $0 \leq \alpha < 1$. In the main result of this section (Proposition 3.1), we show that $u \in C^{2,1}(\mathcal{C})$.

Thanks to (3.1), the infinitesimal generator \mathcal{L} can be rewritten such that its integral component has a reduced form, i.e., $\mathcal{L}\phi(x, t) = \mathcal{L}_D^f \phi(x, t) + I^f \phi(x, t)$ in which $\mathcal{L}_D^f \phi(x, t) = \sum_{i,j=1}^n a_{ij} \partial_{x^i x^j}^2 \phi + \sum_{i=1}^n \left[b_i - \int_{|y| \leq 1} y^i \nu(dy) \right] \partial_{x^i} \phi$ and

$$(3.2) \quad I^f \phi(t, x) \triangleq \int_{\mathbb{R}^n} [\phi(x + y, t) - \phi(x, t)] \nu(dy).$$

Thanks to this reduced integral form and the Lipschitz continuity of $u(\cdot, t)$ (see Lemma 2.2), $I^f u(x, t)$ is well defined in the class sense. Indeed, it follows from (3.1) and (H2) that $|I^f u(x, t)| \leq \int_{\mathbb{R}} |u(x + y, t) - u(x, t)| \nu(dy) \leq L_x \int_{\mathbb{R}} |y| \nu(dy) < +\infty$. Moreover, $I^f u(x, t)$ is Hölder continuous in its both variables.

Lemma 3.1. *Let Ω be any compact domain in \mathbb{R}^n . Assume that the density $\rho(y)$ satisfies (H5) with $0 \leq \alpha < 1$.*

- (i) *For any $(x_1, t), (x_2, t) \in \Omega \times [0, T]$, there exist constants $C_{\Omega, \beta}$ and C_Ω , independent of x_1, x_2 and t , such that*

$$(3.3) \quad \text{when } \alpha = 0 : \quad |I^f u(x_1, t) - I^f u(x_2, t)| \leq C_{\Omega, \beta} |x_1 - x_2|^{1-\beta}, \quad \text{for any } \beta \in (0, 1);$$

$$(3.4) \quad \text{when } 0 < \alpha < 1 : \quad |I^f u(x_1, t) - I^f u(x_2, t)| \leq C_\Omega |x_1 - x_2|^{1-\alpha}.$$

- (ii) *For any $(x, t_1), (x, t_2) \in \Omega \times [0, T]$, there exist constants $D_{\Omega, \beta}$ and D_Ω , independent of t_1, t_2 and x , such that*

$$(3.5) \quad \text{when } \alpha = 0 : \quad |I^f u(x, t_1) - I^f u(x, t_2)| \leq D_{\Omega, \beta} |t_1 - t_2|^{\frac{1-\beta}{2}}, \quad \text{for any } \beta \in (0, 1);$$

$$(3.6) \quad \text{when } 0 < \alpha < 1 : \quad |I^f u(x, t_1) - I^f u(x, t_2)| \leq D_\Omega |t_1 - t_2|^{\frac{1-\alpha}{2}}.$$

Proof. Our proof is motivated by Proposition 2.5 in [29]. See Appendix A for details. \square

Now let us analyze (2.8) on a given compact domain inside the continuation region \mathcal{C} . Let B be an open ball in \mathbb{R}^n such that $B \times (t_1, t_2) \subset \mathcal{C}$ for some $t_1, t_2 \in [0, T]$. We will denote the closure of B by \overline{B} .

Instead of working with the nonlocal equation (2.8), note that $I^f u$ is well defined in the classical sense, we consider the following local equation with the driving term $I^f u$:

$$(3.7) \quad \begin{aligned} (-\partial_t - \mathcal{L}_D^f + r) v(x, t) &= I^f u(x, t), \quad \text{for } (x, t) \in B \times [t_1, t_2), \\ v(x, t) &= u(x, t), \quad \text{for } (x, t) \in \partial B \times [t_1, t_2) \cup \overline{B} \times t_2. \end{aligned}$$

A viscosity solution of (3.7) is defined as follows. (See e.g. Definition 7.4 in [10], Definition 13.1 in [13].)

Definition 3.1. Any $v \in C^0(\overline{B} \times [t_1, t_2])$ is a viscosity subsolution of (3.7) if

$$(3.8) \quad (-\partial_t - \mathcal{L}_D^f + r) \phi(x, t) \leq I^f u(x, t), \quad \text{for } (x, t) \in B \times [t_1, t_2],$$

$$(3.9) \quad \min \left\{ (-\partial_t - \mathcal{L}_D^f + r) \phi(x, t) - I^f u(x, t), v(x, t) - u(x, t) \right\} \leq 0, \quad \text{for } (x, t) \in \partial B \times [t_1, t_2] \cup \overline{B} \times t_2$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x, t) = v(x, t)$ and $\phi(\tilde{x}, \tilde{t}) \geq v(\tilde{x}, \tilde{t})$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [t_1, t_2]$. The supersolution is defined analogously. As usual, v is a viscosity of (3.7) if it is both a subsolution and a supersolution.

Based on Proposition 2.1 and an equivalence between Definition 3.1 and another notion of viscosity solutions for a nonlocal equation, we have the following result, whose proof is listed in Appendix A.

Lemma 3.2. If the Lévy measure ν satisfies (H2) and (H5) with $0 \leq \alpha < 1$, then u is a viscosity solution of (3.7).

Now let us assume that the diffusion component of X is nondegenerate i.e., there exists $\lambda > 0$ such that

$$(H6) \quad \sum_{i,j=1}^n a_{ij}(x, t) \xi^i \xi^j \geq \lambda |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^n, t \geq 0.$$

Additionally, we assume

$$(H7) \quad a_{ij}(x, t), b_i(x, t) \text{ and } r(x, t) \text{ are continuously differentiable in both variables on } \mathbb{R}^n \times [0, T] \text{ for } i, j \leq n.$$

We are ready to show the regularity of u inside the continuation region.

Proposition 3.1. Assume that the Lévy measure ν satisfies (H2) and (H5) with $0 \leq \alpha < 1$, moreover (H6) and (H7) are satisfied. Then $u \in C^{2,1}(\mathcal{C})$.

Proof. First it follows from Lemma 2.2 that the boundary and terminal values of (3.7) are continuous on $\partial B \times [t_1, t_2] \cup \overline{B} \times t_2$. Moreover, the driving term $I^f u(x, t)$ is uniformly Hölder continuous in $\overline{B} \times [t_1, t_2]$ thanks to Lemma 3.1. Therefore, (3.7), whose coefficients satisfying (H6) and (H7), has a unique classical solution $u^*(x, t) \in C^{2,1}(B \times (t_1, t_2))$ (see Theorem 9 in [14] pp. 69). Since u^* is already a classical solution, u^* is also a viscosity solution of (3.7). Now, it follows from the comparison theorem for viscosity solutions (see e.g. Theorem 7.5 in [10]) that $u(x, t) = u^*(x, t)$ for $(x, t) \in B \times (t_1, t_2)$. Therefore $u \in C^{2,1}(B \times (t_1, t_2))$. The statement follows since $B \times (t_1, t_2)$ is chosen arbitrarily in \mathcal{C} . \square

4. INFINITE VARIATION JUMPS AND THE REGULARITY ON THE WHOLE DOMAIN

In this section, we shall study the regularity of the value function on the whole domain. Moreover, jumps in this section may have infinite variation (i.e., (3.1) may not be satisfied). We give the main result of the paper in Theorem 4.1, which states that $u \in W_p^{2,1}(B)$, for any compact domain $B \subset \mathbb{R}^n \times (0, T)$ and $p > 1$. The proof of this result is given in Section 5. There are two important corollaries to Theorem 4.1: In Corollary 4.1, we show that the smooth fit property holds; in Proposition 4.1 we show that $u \in C^{2,1}(\mathcal{C})$. We start by developing some properties of the integral operator I in Lemma 4.1. These properties will be crucial in our proofs.

4.1. Notation.

Definition 4.1. Let Ω be a domain in \mathbb{R}^n , $Q_T = \Omega \times (0, T)$ and $\overline{Q_T}$ be the closure of Q_T .

$C^{2,1}(Q_T)$ denotes the class of continuous functions v on Q_T with continuous classical derivatives $\partial_t v$, $\partial_{x^i} v$ and $\partial_{x^i x^j}^2 v$ for $i, j \leq n$ on Q_T .

For any positive integer $p \geq 1$, $W_p^{2,1}(Q_T)$ is the Banach space consisting of the elements of $v \in L_p(Q_T)$ having generalized derivatives of the form $\partial_t v$, $\partial_{x^i} v$ and $\partial_{x^i x^j}^2 v$ for $i, j \leq n$ and a finite norm $\|v\|_{W_p^{2,1}(Q_T)}$. (Please refer to pp. 5 of [22] for the definition of $\|\cdot\|_{W_p^{2,1}}$.)

In particular, $W_{p,loc}^{2,1}(Q_T)$ is the Banach space consisting of functions whose $W_p^{2,1}$ -norm is finite on any compact subsets of Q_T .

For any positive nonintegral real number α , $H^{\alpha, \alpha/2}(\overline{Q_T})$ is the Banach space of functions v that are continuous in $\overline{Q_T}$, and have continuous bounded classical derivatives of the form $\partial_t^r \partial_x^s v$ for $2r + s < \alpha$, which satisfy $\|v\|_{\overline{Q_T}}^{(\alpha)} < \infty$. (Please refer to pp. 7 of [22] for the definition of $\|\cdot\|^{(\alpha)}$.)

On the other hand, $H^\alpha(\overline{\Omega})$ is the Banach space whose elements are continuous functions $v(x)$ on $\overline{\Omega}$ that have continuous bounded derivatives up to order $[\alpha]$ and satisfy $\|v\|_{\overline{\Omega}}^{(\alpha)} < \infty$.

4.2. The integral term. When the jumps of X have infinite variation, i.e., (3.1) is not satisfied, the integral term cannot be reduced to the form in (3.2). Therefore, throughout this section we need to work with the integro-differential operator \mathcal{L} and its integral part I in the form of (2.9) and (2.10). However, given the regularity properties of u in Lemma 2.2, it is not clear that Iu is well defined in the classical sense. (See the discussion after (2.10).) Nevertheless, we shall show in the following lemma that given sufficient regularity properties for the test function ϕ , $I\phi(x, t)$ is Hölder continuous in both variables. Later in this section, we will prove that the value function u does have these regularity properties to guarantee Iu well defined in the classical sense.

Let Ω be a compact domain in \mathbb{R}^n , $\Omega^\delta \triangleq \{x \in \mathbb{R}^n : x \in B_\delta(y) \text{ for some } y \in \Omega\}$ for some $\delta > 0$. For $s \in (0, T]$, let us denote $\overline{Q_s} = \overline{\Omega} \times [0, s]$ and $\overline{Q_s^\delta} = \overline{\Omega^\delta} \times [0, s]$. Moreover, we denote $D_s \triangleq \mathbb{R}^n \times [0, s]$.

Lemma 4.1. Let us assume that the Lévy measure satisfies (H2) and (H5) with $\alpha \in [1, 2]$.

- (i) Assume that ϕ satisfies $\max_{\mathbb{R}^n \times [0, s]} |\phi| < \infty$, $\max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| < \infty$, and that there exists $\tilde{L}_t > 0$ such that $|\phi(x, t_1) - \phi(x, t_2)| \leq \tilde{L}_t |t_1 - t_2|^{1/2}$ for any $x \in \mathbb{R}^n$ and $t_1, t_2 \in [0, s]$. If $\phi \in H^{\beta, \frac{\beta}{2}}(\overline{Q_s^\delta})$ for some $\beta \in (\alpha, 2)$, then $Iu \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(\overline{Q_s})$. Additionally, there exists a constant $C_\Omega > 0$, depending on Ω , α , β and T , such that

$$(4.1) \quad \|I\phi\|_{\overline{Q_s}}^{(\frac{\beta-\alpha}{2})} \leq C_\Omega \left(\max_{\mathbb{R}^n \times [0, s]} |\phi| + \max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| + \tilde{L}_t + \|\phi\|_{\overline{Q_s^\delta}}^{(\beta)} \right).$$

- (ii) If $\phi \in H^{\beta, \frac{\beta}{2}}(D_s)$ for some $\beta \in (\alpha, 2)$, then $I\phi \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$. Moreover, there exists a constant C , depending on α, β and T , such that

$$(4.2) \quad \|I\phi\|_{D_s}^{(\frac{\beta-\alpha}{2})} \leq C \|\phi\|_{D_s}^{(\beta)}.$$

Proof. The proof is similar to the proof of Lemma 3.1. See Appendix A. \square

Remark 4.1. When the Lévy measure ν is a finite measure on \mathbb{R}^n , the integral form $\int_{\mathbb{R}^n} \phi(x + y, t) \nu(y)$ has the same regularity as $\phi(x, t)$; see [31]. When the Lévy measure has a singularity, as we have seen in Lemma 4.1, the mode of continuity of $I\phi$ decreases compared to the mode of continuity of ϕ . Moreover, as we have seen in (4.1), the Hölder norm of $I\phi$ depends on the Hölder norm of ϕ on a slightly larger domain. This extension of domains will introduce a technical difficulty in estimating the Sobolev norm of u . This estimation will be carried out in the following section.

4.3. Solutions in the Sobolev sense. In this subsection, we shall give the main result of this paper. Compared to Section 3, we need some stronger assumptions on coefficients. Instead of (H7), we assume that

$$(H7') \quad a_{ij}(x, t), b(x, t), r(x, t) \in H^{\ell, \frac{\ell}{2}}(\mathbb{R}^n \times [0, T]), \quad \forall \ell \in (0, 1) \text{ and } i, j \leq n, \text{ and } r(x, t) \geq 0.$$

Moreover, there exist positive constants λ such that

$$(H6') \quad \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall (x, t) \in \mathbb{R}^n \times [0, T] \text{ and } \xi \in \mathbb{R}^n.$$

In addition to (H3) and (H4), we assume that there exists a positive constant J such that

$$(H8) \quad \frac{\partial^2}{\partial \eta^2} g \geq -J, \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \quad \text{for any direction } \eta \in \mathbb{R}^n,$$

in which $\partial/\partial \eta$ is the directional derivative, and the inequality is understood in the distributional sense.

Let ζ^ϵ be the standard mollifier (see [7] pp. 629 for its definition and properties). Consider the mollified sequence $g^\epsilon \triangleq g * \zeta^\epsilon$ for $\epsilon \in (0, \epsilon_0)$. Here ϵ_0 is a positive constant less than 1. First, it follows from (H8) that

$$(4.3) \quad \sum_{i,j=1}^n \partial_{x^i x^j}^2 g^\epsilon(x) \xi^i \xi^j \geq -J |\xi|^2, \quad \forall \xi, x \in \mathbb{R}^n.$$

It is clear that

$$(4.4) \quad \text{each } g^\epsilon(x) \in H^{2+\ell}(\mathbb{R}^n) \quad \forall \ell \in (0, 1).$$

Additionally, (H3) and (H4) imply that there exist positive constants K and L independent of ϵ such that for all $x \in \mathbb{R}^n$

$$(4.5) \quad 0 \leq g^\epsilon(x) \leq K,$$

$$(4.6) \quad |\nabla g^\epsilon(x)| \leq L.$$

Now we are ready to state main result of this paper.

Theorem 4.1. *If (H3), (H4), (H6), (H7'), and (H8) are satisfied, and the Lévy measure ν satisfies (H2) and (H5) with $\alpha \in [0, 2)$, then $u \in W_p^{2,1}(B_\rho(x_0) \times (0, T-s))$ for any integer $p \in (1, \infty)$, $\rho, s \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$. Moreover, u solves (2.8) for almost every point in $\mathbb{R}^n \times [0, T]$.*

Before we prove this result in Section 5, let us list some of its corollaries.

Corollary 4.1. *If the assumptions in Theorem 4.1 are satisfied, then for any $\rho, s > 0$ and $x_0 \in \mathbb{R}^n$*

- (i) *$u \in H^{\beta, \frac{\beta}{2}}(\overline{B_\rho(x_0)} \times [0, T-s])$ where $\beta = 2 - \frac{n+2}{p} > 0$. In particular, $\nabla_x u \in C(\mathbb{R}^n \times [0, T])$. Therefore, the smooth-fit property holds.*
- (ii) *If the Lévy measure ν satisfies (H5) with $\alpha \in [1, 2)$, then Iu is well defined in the classical sense in $B_\rho(x_0) \times [0, T]$. Moreover, $Iu \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(\overline{B_\rho(x_0)} \times [0, T-s])$ for some $\beta \in (\alpha, 2)$.*

Proof. (i) Combining Theorem 4.1 and the Sobolev Inequality (see e.g. Lemma 3.3 in [22] pp. 80), we have $u \in H^{\beta, \frac{\beta}{2}}(\overline{B_\rho(x_0)} \times [0, T-s])$, where $\beta = 2 - \frac{n+2}{p} > 0$. Choosing sufficiently large p such that $\beta > 1$, the continuity of $\nabla_x u$ follows from Definition 4.1.

- (ii) Let us choose p sufficiently large so that $\beta > \alpha$. Now, the proof follows from (i) and Lemma 4.1. \square

Thanks to Corollary 4.1 (ii), we can consider the following boundary value problem with the driving term Iu :

$$(4.7) \quad \begin{aligned} (-\partial_t - \mathcal{L}_D + r)v(x, t) &= Iu(x, t), \quad \text{for } (x, t) \in B \times [t_1, t_2], \\ v(x, t) &= u(x, t), \quad \text{for } (x, t) \in \partial B \times [t_1, t_2] \cup \overline{B} \times t_2, \end{aligned}$$

where $B \times (t_1, t_2) \subset \mathcal{C}$ is the bounded domain as in (3.7). The viscosity solution of (4.7) is defined as in Definition 3.1, after replacing \mathcal{L}_D^f and I^f by \mathcal{L}_D and I , respectively.

The following relation between the solutions in the Sobolev sense and the viscosity sense shows that u is a viscosity solution of (4.7). See Corollary 3 in [25] or Theorem 9.15 (ii) in [20] for its proof.

Lemma 4.2. *If $u \in W_p^{2,1}(B \times (t_1, t_2))$ for $p > n + 1$ satisfies (4.7) at almost every point in $B \times (t_1, t_2)$, then u is the viscosity solution of (4.7) in the sense of Definition 3.1.*

Thanks to Corollary 4.1, Lemmas 4.1 and 4.2, the arguments in the proof of Proposition 3.1 now works for the infinite variation jump case.

Proposition 4.1. *If the assumptions of Theorem 4.1 are satisfied, then $u \in C^{2,1}(\mathcal{C})$.*

Proof. Corollary 4.1 (ii) tells us that $Iu(x, t)$ is Hölder continuous in both its variable. Moreover, u is a viscosity solution of (4.7) thanks to Lemma 4.2. The rest proof follows the same line of arguments in the proof of Proposition 3.1. \square

In our future work, we will investigate the regularity of the free boundary curves and extend our results in [5].

5. PROOF OF THEOREM 4.1

Because the jump may have infinite variation, the proof of Theorem 4.1 needs to conquer several technical difficulties. We will carry the proof of Theorem 4.1 in a series of lemmas and point out the difficulties along the way.

Let us first define $v(x, t) = u(x, T - t)$ for $(x, t) \in \mathbb{R}^n \times [0, T]$. It is natural to expect that v solves the following variational inequality

$$(5.1) \quad \begin{aligned} \min \{(\partial_t - \mathcal{L}_D - I + r)v(x, t), v(x, t) - g(x)\} &= 0, \quad (x, t) \in \mathbb{R}^n \times (0, T], \\ v(x, 0) &= g(x). \end{aligned}$$

We will establish Theorem 4.1 by using the penalty method, which constructs a sequence of approximating functions each of which solves (5.2). First, in Lemma 5.2, we find a nice enough solution, v^ϵ , to each penalty problem. In Corollary 5.1 we give a uniqueness result for each of these penalty problems. Second, we analyze the properties of the value functions of the penalty problems in Lemmas 5.4, 5.5, 5.6, and Corollary 5.2. These are used to show that the $W_p^{2,1}$ -norm of v^ϵ is bounded uniformly in ϵ in Corollary 5.3. In order to establish the latter result, we also prove a $W_p^{2,1}$ -norm estimate for the solutions of parabolic integro-differential equations in Theorem 5.1. We show in Corollary 5.4 that the weak limit of $\{v^\epsilon\}$, which we denote by v^* , solves the variational inequality and has a finite $W_p^{2,1}$ norm. This result along with Proposition 5.1 concludes the proof of Theorem 4.1.

In the following, we will only carry out the proof of Theorem 4.1 for the infinite variation jump case, i.e., the Lévy measure ν satisfies (H5) with $1 \leq \alpha < 2$. Since the integral operator has the reduced form I^f in (3.2) for the finite variation jumps, the proof of $0 \leq \alpha < 1$ case in Theorem 4.1 will be similar and easier.

Motivated by Lemma 3.1 in [15] pp. 24 and [31], we will study the following penalty problem for each $\epsilon \in (0, \epsilon_0)$:

$$(5.2) \quad \begin{aligned} (\partial_t - \mathcal{L}_D - I + r)v^\epsilon(x, t) + p_\epsilon(v^\epsilon - g^\epsilon) &= 0, \quad (x, t) \in \mathbb{R}^n \times (0, T], \\ v^\epsilon(x, 0) &= g^\epsilon(x), \end{aligned}$$

in which $\{g^\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ is given after (H8). Here, the penalty term $p_\epsilon(y) \in C^\infty(\mathbb{R})$ is chosen to satisfy following properties:

(5.3)

$$(i) p_\epsilon(y) \leq 0, \quad (ii) p_\epsilon(y) = 0 \text{ if } y \geq \epsilon, \quad (iii) p_\epsilon(0) = -n\Lambda J - |b|^{(0)}L - |r|^{(0)}K - J \int_{|y| \leq 1} |y|^2 \nu(dy) - K \int_{|y| > 1} \nu(dy),$$

$$(iv) p'_\epsilon(y) \geq 0, \quad (v) p''_\epsilon(y) \leq 0 \quad \text{and} \quad (vi) \lim_{\epsilon \downarrow 0} p_\epsilon(y) = \begin{cases} 0, & y > 0 \\ -\infty, & y < 0 \end{cases}.$$

The constants Λ, K, L and J come from (H6), (4.5), (4.6), and (H8), respectively. Additionally, $|b|^{(0)} = \max_{\mathbb{R}^n \times [0, T]} |b(x, t)|$ and $|r|^{(0)} = \max_{\mathbb{R}^n \times [0, T]} |r(x, t)|$ are finite due to (H7'). Moreover, $p_\epsilon(0)$ is also finite thanks to (2.3). It is also worth pointing out that $p_\epsilon(0)$ is independent of ϵ . These properties of p_ϵ will be useful in the development of our next few results. In particular, (5.3) (iii) is essential for proofs of Lemma 5.6 and Corollary 5.2.

Let us recall the Schauder fixed point theorem (see e.g. Theorem 2 in [14] pp. 189).

Lemma 5.1. *Let Θ be a closed convex subset of a Banach space and let \mathcal{T} be a continuous operator on Θ such that $\mathcal{T}\Theta$ is contained in Θ and $\mathcal{T}\Theta$ is precompact. Then \mathcal{T} has a fixed point in Θ .*

For each $\epsilon \in (0, \epsilon_0)$, we will show that the penalty problem (5.2) has a classical solution via the Schauder fixed point theorem. Let us recall $D_s = \mathbb{R}^n \times [0, s]$.

Lemma 5.2. *If the Lévy measure ν satisfies (H2) and (H5) with $1 \leq \alpha < 2$, then for any $\epsilon \in (0, \epsilon_0)$ and $\beta \in (\alpha, 2)$, (5.2) has a solution $v^\epsilon \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_T)$.*

Proof. We will first prove that (5.2) has a solution on a sufficiently small time interval $t \in [0, s]$ via the Schauder fixed point theorem. Then we will extend this solution to the interval $[0, T]$.

Let us consider the set $\Theta = \left\{ v \in H^{\beta, \frac{\beta}{2}}(D_s) \text{ with its Hölder norm } \|v\|_{D_s}^{(\beta)} \leq U_0 \right\}$, where positive constants s and U_0 will be determined later. It is clear that Θ is a bounded, closed and convex set in the Banach space $H^{\beta, \frac{\beta}{2}}(D_s)$. For any $v \in \Theta$, consider the following Cauchy problem for $u - g^\epsilon$:

$$(5.4) \quad \begin{aligned} (\partial_t - \mathcal{L}_D + r)(u - g^\epsilon)(x, t) &= Iv(x, t) - p_\epsilon(v - g^\epsilon)(x, t) + (\mathcal{L}_D - r)g^\epsilon(x), \quad (x, t) \in \mathbb{R} \times (0, s], \\ u(x, 0) - g^\epsilon(x) &= 0. \end{aligned}$$

Via the solution u of (5.4), the operator \mathcal{T} can be defined as $u = \mathcal{T}v$. Let us check the conditions for the Schauder fixed point theorem are satisfied:

1. $\mathcal{T}v$ is well defined. Since $v \in H^{\beta, \frac{\beta}{2}}(D_s)$ and $\beta \in (\alpha, 2)$, it follows from Lemma 4.1 (ii) that $Iv \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$ with

$$(5.5) \quad \|Iv\|_{D_s}^{(\frac{\beta-\alpha}{2})} \leq C \|v\|_{D_s}^{(\beta)}, \quad \text{for some constant } C > 0 \text{ independent of } s.$$

On the other hand, we can check that $p_\epsilon(v - g^\epsilon) \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$. Indeed, $p_\epsilon(v - g^\epsilon)$ is bounded in D_s , since both $v, g^\epsilon \in H^{\beta, \frac{\beta}{2}}(D_s)$ (see (4.4)) and $p_\epsilon(y) \in C^0(\mathbb{R})$. Additionally, for any $x_1, x_2 \in \mathbb{R}^n, t \in [0, s]$

$$|p_\epsilon(v - g^\epsilon)(x_1, t) - p_\epsilon(v - g^\epsilon)(x_2, t)| \leq \max_{D_s} |p'_\epsilon(v - g^\epsilon)| |(v - g^\epsilon)(x_1, t) - (v - g^\epsilon)(x_2, t)| \leq \tilde{C} |x_1 - x_2|.$$

Here $\max_{D_s} |p'_\epsilon(v - g^\epsilon)|$ is finite, which also follows from the boundness of $v - g^\epsilon$ and $p_\epsilon \in C^1(\mathbb{R})$. The positive constant \tilde{C} depends on $\max_{D_s} |p'_\epsilon(v - g^\epsilon)|$ and the Hölder norms of v and g^ϵ . The Hölder continuity of $p_\epsilon(v - g^\epsilon)$ in t can be checked similarly. Furthermore, $(\mathcal{L}_D - r)g^\epsilon(x) \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$ as a result of (4.4). Therefore, thanks to (H6) and (H7'), it follows from Theorem 5.1 in [22] pp. 320 that (5.4) has a unique solution $u - g^\epsilon \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_s)$. Note that $g^\epsilon \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_s)$ (see (4.4)). As a result $u = \mathcal{T}v \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_s)$.

2. $\mathcal{T}\Theta \subset \Theta$. For $u = Tv$, appealing to Lemma 2 in [14] pp. 193, we obtain that there exists a positive constant A_β , depending on β , such that

$$(5.6) \quad \begin{aligned} \|u - g^\epsilon\|_{D_s}^{(\beta)} &\leq A_\beta s^\gamma \left[\|Iv\|^{(0)} + \|p_\epsilon(v - g^\epsilon)\|^{(0)} + \|(\mathcal{L}_D - r)g^\epsilon\|^{(0)} \right] \\ &\leq A_\beta C s^\gamma \|v\|_{D_s}^{(\beta)} + \tilde{A}, \end{aligned}$$

where $\gamma = \frac{2-\beta}{2}$, C is the constant in (5.5) and \tilde{A} is a sufficiently large constant dependent on $\|g^\epsilon\|_{\mathbb{R}^n}^{(2+\ell)}$ for some $\ell \in (0, 1)$. Let s be such that $\tau \triangleq A_\beta C s^\gamma < 1/2$ and let $U_0 \triangleq \max\{\frac{2\tilde{A}}{1-2\tau}, 2\|g^\epsilon\|_{D_s}^{(\beta)}\}$. Note that $\|v\|_{D_s}^{(\beta)} \leq U_0$. Now it follows from (5.6) that

$$(5.7) \quad \|u\|_{D_s}^{(\beta)} \leq \|u - g^\epsilon\|_{D_s}^{(\beta)} + \|g^\epsilon\|_{D_s}^{(\beta)} \leq \tau U_0 + \tilde{A} + \frac{U_0}{2} \leq \tau U_0 + \frac{1-2\tau}{2} U_0 + \frac{U_0}{2} = U_0.$$

Therefore, $u = \mathcal{T}v \in \Theta$.

3. $\mathcal{T}\Theta$ is a precompact subset of $H^{\beta, \frac{\beta}{2}}(D_s)$. For any $\eta \in (\beta, 2)$, similar estimate as (5.6) shows that for any $v \in \Theta$, we have $\|Tv\|_{D_s}^{(\eta)} \leq U_1$ for some constant U_1 depending on U_0 and s . On the other hand, argument similar to Theorem 1 in [14] pp.188 shows that bounded subsets of $H^{\eta, \frac{\eta}{2}}(D_s)$ are precompact subsets of $H^{\beta, \frac{\beta}{2}}(D_s)$. Therefore, $\mathcal{T}\Theta$ is a precompact subset in $H^{\beta, \frac{\beta}{2}}(D_s)$.

4. \mathcal{T} is a continuous operator. Let v_n be a sequence in Θ such that $\lim_{n \rightarrow \infty} \|v_n - v\|_{D_s}^{(\beta)} = 0$, we will show $\lim_{n \rightarrow \infty} \|Tv_n - Tv\|_{D_s}^{(\beta)} = 0$. From (5.4), $w \triangleq Tv_n - Tv$ satisfies the Cauchy problem

$$\begin{aligned} (\partial_t - \mathcal{L}_D + r)w(x, t) &= I(v_n - v)(x, t) - [p_\epsilon(v_n - g^\epsilon) - p_\epsilon(v - g^\epsilon)], \quad (x, t) \in \mathbb{R}^n \times (0, s] \\ w(x, 0) &= 0. \end{aligned}$$

It follows again from Lemma 2 in [14] pp. 193 that

$$\begin{aligned} \|Tv_n - Tv\|_{D_s}^{(\beta)} &= \|w\|_{D_s}^{(\beta)} \leq A_\beta s^\gamma \left[\|I(v_n - v)\|^{(0)} + \|p_\epsilon(v_n - g^\epsilon) - p_\epsilon(v - g^\epsilon)\|^{(0)} \right] \\ &\leq A_\beta s^\gamma \left[C\|v_n - v\|_{D_s}^{(\beta)} + \max_{D_s, n} |p'_\epsilon(v_n - g^\epsilon)| \|v_n - v\|^{(0)} \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

As a result of Steps **2.** - **4.** and the Schauder fixed point theorem, we obtain a fixed point of the operator \mathcal{T} in $H^{\beta, \frac{\beta}{2}}(D_s)$. We denote this fixed point by v^ϵ . Moreover, it follows from the result in Step **1** that $v^\epsilon = \mathcal{T}v^\epsilon \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_s)$.

Finally, let us extend v^ϵ to the interval $[0, T]$. Choosing any $\rho \in (0, T - s)$, we replace $g^\epsilon(\cdot)$ by $v^\epsilon(\cdot, \rho)$ in (5.4). Note that the choice of s in **2.** only depends on β and C , but not on ρ . If $\|v^\epsilon(\cdot, \rho)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$ is finite, we can choose a sufficiently large U_0 , depending on $\|v^\epsilon(\cdot, \rho)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$, such that (5.7) holds on $[\rho, \rho + s]$, moreover $\|v^\epsilon(\cdot, \rho + s)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$ is finite thanks to the result after **4.** Noticing that $\|g^\epsilon\|_{\mathbb{R}^n}^{(2+\ell)}$ is finite for any $\ell \in (0, 1)$, one can extend the time interval by s each time, until the time interval contains $[0, T]$. Therefore we have the statement of the lemma. \square

Thanks to the definition of the Hölder spaces, Lemma 5.2 also tells us that v^ϵ is bounded in D_T . In order to show that v^ϵ is the unique bounded classical solution of the penalty problem (5.2), we need the following maximum principle for the parabolic integro-differential operator. The proof of it is provided in Appendix A. (See Lemma 2.1 of [31] for a similar maximum principle, where ν is assumed to be a finite measure on \mathbb{R} .)

Lemma 5.3. *Let us assume that $a_{ij}(x, t)$, $b_i(x, t)$ and $c(x, t)$ are bounded in $\mathbb{R}^n \times [0, T]$ with $A = (a_{ij})_{n \times n}$ satisfying $\sum_{i,j=1}^n a_{ij}(x, t) \xi^i \xi^j > 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$, moreover $c(x, t) \geq 0$ and the Lévy measure satisfies (H2). If $v \in C^0([0, T] \times \mathbb{R}^n) \cap C^{2,1}((0, T] \times \mathbb{R}^n)$ satisfies $(\partial_t - \mathcal{L}_D - I + c(x, t))v(x, t) \geq 0$ in $\mathbb{R} \times (0, T]$ and there exists a*

sufficiently large positive constant m such that $v(x, t) \geq -m$ for $(x, t) \in \mathbb{R}^n \times [0, T]$. Then $v(x, 0) \geq 0$ implies that $v(x, t) \geq 0$ for $(x, t) \in \mathbb{R}^n \times [0, T]$.

As a corollary of this maximum principle, the bounded classical solution of the penalty problem (5.2) is unique.

Corollary 5.1. *For each $\epsilon \in (0, \epsilon_0)$, the penalty problem (5.2) has a unique bounded classical solution.*

Proof. Let us assume v_1 and v_2 are two bounded solutions of (5.2). Then $v_1 - v_2$ satisfies

$$(5.8) \quad \begin{aligned} (\partial_t - \mathcal{L}_D - I + r)(v_1 - v_2) + p_\epsilon(v_1 - g^\epsilon) - p_\epsilon(v_2 - g^\epsilon) &= 0, \quad (x, t) \in \mathbb{R}^n \times (0, T], \\ (v_1 - v_2)(x, 0) &= 0 \end{aligned}$$

On the other hand, it follows from the mean value theorem that $p_\epsilon(v_1 - g^\epsilon) - p_\epsilon(v_2 - g^\epsilon) = p'_\epsilon(y)(v_1 - v_2)$ for some $y \in \mathbb{R}^n$. Moreover, $p'_\epsilon(y)$ is bounded, say by M , thanks to the fact that $p_\epsilon \in C^1(\mathbb{R})$ and v_1, v_2 and g^ϵ are all bounded. Now applying Lemma 5.3 to the equation (5.8) and choosing $c = r + M \geq 0$ (see (5.3) (iv)), we have $v_1(x, t) \geq v_2(x, t)$ for $(x, t) \in \mathbb{R}^n \times (0, T]$. The other direction of the inequality follows from applying the same argument to $v_2 - v_1$. \square

Applying Lemma 5.3, we will analyze some universal properties of v^ϵ for all $\epsilon \in (0, \epsilon_0)$ in the following three lemmas.

Lemma 5.4.

$$0 \leq v^\epsilon(x, t) \leq K + 1, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, T].$$

Proof. Since the proof is similar to the proof of Lemma 2.2 in [31], we give it in the Appendix A. \square

Lemma 5.5.

$$|\partial_{x^k} v^\epsilon(x, t)| \leq L, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, T], 1 \leq k \leq n.$$

Proof. The proof is similar to the proof of Lemma 2.4 in [31]. \square

Lemma 5.6. *For any $\epsilon \in (0, \epsilon_0)$, $v^\epsilon(x, t) \geq g^\epsilon(x)$ on $\mathbb{R}^n \times [0, T]$.*

Proof. Let us first show that $Ig^\epsilon(x)$ is uniformly bounded from below. Indeed,

$$(5.9) \quad \begin{aligned} Ig^\epsilon(x) &= \int_{|y| \leq 1} \left[g^\epsilon(x + y) - g^\epsilon(x) - \sum_{i=1}^n y^i \frac{\partial}{\partial x^i} g^\epsilon(x) \right] \nu(dy) + \int_{|y| > 1} [g^\epsilon(x + y) - g^\epsilon(x)] \nu(dy) \\ &= \int_{|y| \leq 1} \nu(dy) \int_0^1 dz (1 - z) \sum_{i,j=1}^n y^i y^j \frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x + zy) + \int_{|y| > 1} [g^\epsilon(x + y) - g^\epsilon(x)] \nu(dy) \\ &\geq \int_{|y| \leq 1} \nu(dy) \int_0^1 dz (1 - z) (-J|y|^2) - K \int_{|y| > 1} \nu(dy) \\ &\geq -J \int_{|y| \leq 1} |y|^2 \nu(dy) - K \int_{|y| > 1} \nu(dy), \end{aligned}$$

where the first inequality follows from (H8) and (4.5).

On the other hand, thanks to (H6) and (H8), $\sum_{i,j} a_{ij}(x, t) \partial_{x^i x^j}^2 g^\epsilon(x)$ is also bounded from below. Note that $\sum_{i,j} a_{ij}(x, t) \partial_{x^i x^j}^2 g^\epsilon(x) = \text{tr}(AH(g^\epsilon))$, where $H(g^\epsilon)$ is the Hessian of g^ϵ , i.e., $H(g^\epsilon)_{ij} = \partial_{x^i x^j}^2 g^\epsilon(x)$. It follows from the first inequality in (H6) that A is a positive definite matrix. Then there exists a nonsingular matrix C such that $A = CC'$. Therefore $\text{tr}(AH(g^\epsilon)) = \text{tr}(CC'H(g^\epsilon)) = \text{tr}(C'H(g^\epsilon)C)$. Moreover, (H8) and (H6) give us that

$$(C\xi)' H(g^\epsilon) (C\xi) \geq -J (\xi' C' C \xi) = -J (\xi' A \xi) \geq -J \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Hence $C' H(g^\epsilon) C + J\Lambda I_n$ is a non-negative definite matrix. As a result, we have $\text{tr} \left(C' H(g^\epsilon) C \right) + nJ\Lambda = \text{tr} \left(C' H(g^\epsilon) C + J\Lambda I_n \right) \geq 0$, which implies

$$(5.10) \quad \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x) = \text{tr} (AH(g^\epsilon)) \geq -nJ\Lambda.$$

Thanks to (5.9) and (5.10), we can bound $(\partial_t - \mathcal{L}_D - I + r) g^\epsilon(x)$ from above. Indeed,

$$(5.11) \quad \begin{aligned} & (\partial_t - \mathcal{L}_D - I + r) g^\epsilon(x) \\ &= - \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x) - \sum_{i=1}^n b_i(x, t) \frac{\partial}{\partial x^i} g^\epsilon(x) + r(x, t) g^\epsilon(x) - I g^\epsilon(x) \\ &\leq nJ\Lambda + |b|^{(0)} L + |r|^{(0)} K + J \int_{|y| \leq 1} |y|^2 \nu(dy) + K \int_{|y| > 1} \nu(dy) \\ &= -p_\epsilon(0), \end{aligned}$$

where the second equality follows from (5.3) (iii).

Now we will show that $v^\epsilon \geq g^\epsilon$ using Lemma 5.3. It follows from (5.11) that

$$\begin{aligned} (\partial_t - \mathcal{L}_D - I + r) (v^\epsilon - g^\epsilon) &= -p_\epsilon (v^\epsilon - g^\epsilon) - (\partial_t - \mathcal{L}_D - I + r) g^\epsilon \\ &\geq -p_\epsilon (v^\epsilon - g^\epsilon) + p_\epsilon(0). \end{aligned}$$

The last equation together with the mean value theorem implies that

$$(5.12) \quad \left(\partial_t - \mathcal{L}_D - I + r + p'_\epsilon(y) \right) (v^\epsilon - g^\epsilon) \geq 0,$$

for some $y \in \mathbb{R}^n$. Therefore the statement of the lemma follows applying Lemma 5.3 to (5.12) and choosing $c = r + p'_\epsilon(y) \geq 0$. \square

As an easy corollary, the penalty terms are uniformly bounded.

Corollary 5.2. $p_\epsilon (v^\epsilon - g^\epsilon)$ is bounded uniformly in $\epsilon \in (0, \epsilon_0)$.

Proof. Thanks to Lemma 5.6 and (5.3) (i) and (iv), we have $p_\epsilon(0) \leq p_\epsilon (v^\epsilon - g^\epsilon) \leq 0$. The statement follows noticing that $p_\epsilon(0)$ (in (5.3) (iii)) is independent of ϵ . \square

Thanks to Lemmas 5.2, 5.4, 5.5 and Corollary 5.2, we can apply the following $W_p^{2,1}$ -norm estimate for the parabolic integro-differential equation to each solution v^ϵ of the penalty problem.

Since the proof of the following theorem is technical and independent of the penalty problem, we will perform it in Section 6.

Theorem 5.1. *Let us assume the Lévy measure satisfies (H5) with $\alpha \in [0, 2)$, if v is a $W_{p,loc}^{2,1}$ solution of the following Cauchy problem for some positive integer p ,*

$$(5.13) \quad \begin{aligned} & (\partial_t - \mathcal{L}_D - I + r) v = f(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, T], \\ & v(x, 0) = g(x), \end{aligned}$$

where the coefficients satisfy (H6), (H7') and $f \in L_{p,loc}(\mathbb{R}^n \times (0, T))$, moreover $|v|$ is bounded on $\mathbb{R}^n \times [0, T]$ and $|\nabla_x v|$ is bounded on any compact domain of $\mathbb{R}^n \times [0, T]$. Then for any domain $B_\rho(x_0) \times (s, T)$ with $\rho > 0$, $s \in (0, T)$ and $x_0 \in \mathbb{R}^n$

$$(5.14) \quad \|v\|_{W_p^{2,1}(B_\rho(x_0) \times (s, T))} \leq C_\delta \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\delta/4+1}(x_0) \times [0, T]} |\nabla_x v| + \|f\|_{L^p(B_{\rho+\delta/4}(x_0) \times (\delta/2, T))} \right],$$

for some positive constant C_δ and $\delta < s$.

Remark 5.1. The existence of the $W_p^{2,1}$ solution for (5.13) was ensured by Theorem 3.2 in [6] pp.234. However, the norm estimation was not given there. On the other hand, since the integral operator I is non-local, it is important to study the Cauchy problem (5.13) on the entire domain $\mathbb{R}^n \times [0, T]$. Otherwise, for the Cauchy problem on bounded domains of $\mathbb{R}^n \times [0, T]$ with some boundary conditions, $W_p^{2,1}$ solutions are not expected in general, see [17] for a counterexample.

A $W_p^{2,1}$ -norm estimate, similar to (5.14), for the parabolic integro-differential equation was proved in Theorem 3.5 in [16] pp. 91. However, the estimation in [16] requires the jump restricted in a bounded domain, i.e., if $x \in \Omega$ where Ω is a bounded domain in \mathbb{R}^n , the jump size $z(x)$, which is state dependent, can only be chosen such that $x + z(x) \in \Omega$ (see (1.54) in [16] pp. 63). However, this restriction is not satisfied in our case, where the jump size is unbounded and independent of the state variable x .

Applying Theorem 5.1 to each penalty problem (5.2), thanks to Lemmas 5.2, 5.4, 5.5 and Corollary 5.2, we have the following corollary.

Corollary 5.3. If the assumption of Theorem 4.1 are satisfied, then for any domain $B_\rho(x_0) \times (s, T)$ with $\rho > 0$, $s \in (0, T)$ and $x_0 \in \mathbb{R}^n$, $\|v^\epsilon\|_{W_p^{2,1}(B_\rho(x_0) \times (s, T))}$ are bounded uniformly in $\epsilon \in (0, \epsilon_0)$ for any integer $p \in (1, \infty)$, i.e., there is a constant C independent of ϵ such that

$$(5.15) \quad \|v^\epsilon\|_{W_p^{2,1}(B_\rho(x_0) \times (s, T))} \leq C.$$

Proof. It follows from Lemma 5.2 that $v^\epsilon \in W_{p,loc}^{2,1}(\mathbb{R}^n \times (0, T))$. Thanks to Lemmas 5.4 and 5.5, both $\max_{\mathbb{R}^n \times [0, T]} |v^\epsilon|$ and $\max_{\mathbb{R}^n \times [0, T]} |\nabla_x v^\epsilon|$ are bounded uniformly in ϵ . Moreover, it follows from Corollary 5.2 that $f = -p_\epsilon(v^\epsilon - g^\epsilon)$ is also bounded uniformly in ϵ . Therefore, (5.15) follows from (5.14). \square

Remark 5.2. Theorem 5.1 is essential for the proof of Corollary 5.3. However, having infinite variation jumps presents two technical difficulties to the proof of Theorem 5.1. First, as we shall see in Lemma 6.1, once the Lévy measure has a singularity, the L_p -norm of Iv^ϵ depends on the $W_p^{2,1}$ -norm of v^ϵ . Therefore, one could not consider Iv^ϵ as a driving term directly and use the classical $W_p^{2,1}$ -norm estimate for parabolic differential equations (without the integral term) to bound the $W_p^{2,1}$ -norm of v^ϵ by the L_p -norm of Iv^ϵ . On the other hand, when the Lévy measure is a finite measure as in [31], the L_p -norm of Iv^ϵ only depends on the L^∞ -norm of v^ϵ . Therefore, Lemma 2.6 in [31] follows from the classical $W_p^{2,1}$ -norm estimate for parabolic differential equations, i.e., the $W_p^{2,1}$ -norm of v^ϵ is bounded by the L^∞ -norm of v^ϵ .

Second, as we have seen in Remark 4.1 and we shall see it again in Lemma 6.1, the regularity of Iv^ϵ actually depends on regularity of v^ϵ on a larger domain. This extension of the domain is another technical difficulty we face in the proof of Theorem 5.1, because the extension of domains implies that $W_p^{2,1}$ -norm of v^ϵ on a bounded domains depends on its $W_p^{2,1}$ -norm on a slightly larger domain.

To conclude this section, in the following theorem we will find a limit v^* of the sequence $\{v^\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ and show that it is the value function v defined at the beginning of this section.

Corollary 5.4. Let us assume that the assumptions we made in Theorem 4.1 are satisfied. Then for any $s, \rho > 0$ and $x_0 \in \mathbb{R}^n$, there exists a subsequence $\{\epsilon_k\}_{k \geq 0}$ such that v^{ϵ_k} converges uniformly to the limit v^* uniformly in $\overline{B_\rho(x_0)} \times [s, T]$ as $\epsilon_k \rightarrow 0$. Moreover, v^* solves the variational inequality (5.1) for almost every point in $\mathbb{R}^n \times [0, T]$ and $v^* \in W_p^{2,1}(B_\rho(x_0) \times (s, T))$ for any integer $p \in (1, \infty)$.

Proof. Combining Corollary 5.3 and the fact that $W_p^{2,1}$ is weakly compact, we can find a subsequence $\{\epsilon_k\}$ with $\epsilon_k \rightarrow 0$ and a function $v^* \in W_p^{2,1}(B_\rho(x_0) \times (s, T))$ such that

$$(5.16) \quad v^{\epsilon_k} \rightharpoonup v^* \quad \text{in } W_p^{2,1}(B_\rho(x_0) \times (s, T)).$$

Here “ \rightharpoonup ” represents weak convergence. Refer to Appendix D.4. in [12] pp. 639 for its definition and properties. The rest of the proof is the same as proof of Theorem 3.2 in [31]. It confirms that v^* solves the variational inequality (5.1) for almost every point in $\mathbb{R}^n \times [0, T]$. \square

Next, we present a verification theorem relying on the stability and uniqueness of viscosity solutions adapting the arguments in the proof of Theorem 1 in [3].

Proposition 5.1. *The function $u^*(x, t) = v^*(x, T - t)$ is equal to the value function u .*

Proof. We will show that u^* is a viscosity solution of (2.8). Then the fact that $u^* = u$ is a consequence of Proposition 2.1.

First, let us show that u^* is a viscosity subsolution of (2.8). Consider a point (x, t) and a test function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap C_1(\mathbb{R}^n \times [0, T])$ such that $u^*(\tilde{x}, \tilde{t}) \leq \phi(x, t) = u^*(x, t)$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [0, T]$.

Let us assume that $u^*(x, t) > g(x)$, otherwise there is nothing to prove. Therefore, we need to prove that

$$(\partial_t - \mathcal{L}_D - I + r) \phi(x, t) \leq 0,$$

for such a point (x, t) . In fact, it is enough to prove that for any test function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap C_1(\mathbb{R}^n \times [0, T])$ such that $u^*(\tilde{x}, \tilde{t}) \leq \phi(x, t) = u^*(x, t)$ for any $(\tilde{x}, \tilde{t}) \in B(x, t; \delta)$.

$$(5.17) \quad (\partial_t \phi - \mathcal{L}_D \phi - I_\delta \phi - I^\delta[u^*, \phi] + r\phi)(x, t) \leq 0,$$

in which I_δ is defined in (2.13) and

$$I^\delta[u^*, \phi](x, t) = \int_{|y| > \delta} [u^*(x + y, t) - u^*(x, t)] \nu(dy) - \nabla_x \phi(x, t) \cdot \int_{\delta < |y| \leq 1} y \nu(dy);$$

see e.g. Proposition 1 of [3].

Without loss of generality we can assume that (x, t) is the strict maximum of the function $u^* - \phi$ on $B(t, x; \delta)$, otherwise the test function can be appropriately modified. Then, thanks to the uniform convergence of $u^{\epsilon_k}(x, t) = v^{\epsilon_k}(x, T - t)$ to u^* , and the continuity of the functions in this sequence, we have that for large enough k the function $v^{\epsilon_k} - \phi$ attains its strict maximum over $\bar{B}(x, t; \delta)$ at $(x_k, t_k) \in B(x, t; \delta)$. Moreover, $(x_k, t_k) \rightarrow (x, t)$. Since v^{ϵ_k} is a classical solution of (5.2) (see Corollary 5.1), it is also a viscosity solution. Hence (again using something similar to Proposition 1 of [3]),

$$(5.18) \quad (\partial_t \phi - \mathcal{L}_D \phi - I_\delta \phi - I^\delta[u^{\epsilon_k}, \phi] + r\phi)(x_k, t_k) + p_{\epsilon_k}(u^{\epsilon_k}(x_k) - g^{\epsilon_k}(x_k)) \leq 0.$$

First, since $u^*(x, t) > g(x)$ and that $u^{\epsilon_k}(x_k) - g^{\epsilon_k}(x_k) \rightarrow u^*(x, t) - g(x)$ the penalty term in (5.18) goes to 0. Second, using (2.13) and the dominated convergence theorem we have that $I_\delta \phi(x_k, t_k) \rightarrow I_\delta \phi(x, t)$. Third, using the fact that u^{ϵ_k} is uniformly bounded (see Lemma 5.4) we can exchange the limit and integration in $\lim_{k \rightarrow \infty} I^\delta[u^{\epsilon_k}, \phi](x_k, t_k)$. Furthermore, using the uniform convergence of u^{ϵ_k} to u^* on compacts, we conclude that $I^\delta[u^{\epsilon_k}, \phi](x_k, t_k) \rightarrow I^\delta[u^*, \phi](x, t)$. Therefore, passing to the limit in (5.18) we obtain (5.17).

Second, we will prove that u^* is a viscosity supersolution of (2.8). Since $u^{\epsilon_k} \geq g^\epsilon$ by Lemma 5.6, it follows that $u^* \geq g$. Therefore, we only need to show that

$$(\partial_t - \mathcal{L}_D - I + r) \phi(x, t) \geq 0,$$

for a test function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap C_1(\mathbb{R}^n \times [0, T])$ such that $u^*(\tilde{x}, \tilde{t}) \geq \phi(x, t) = u^*(x, t)$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [0, T)$. The rest of the proof of the supersolution property follows the line of arguments we used for the subsolution property. \square

Theorem 4.1 follows from Corollary 5.4 and Proposition 5.1.

6. PROOF OF THEOREM 5.1

For notational simplicity, the constant C denotes a generic constant in different places. Moreover, the center x_0 of the ball $B_\rho(x_0)$ will not be noted in the sequel. For any positive integer p , let us first estimate the L_p -norm of the integral term Iv .

Lemma 6.1. *If the assumptions of Theorem 5.1 are satisfied, then for any $\eta > 0$, there exists a positive constant C such that*

$$(6.1) \quad \|Iv\|_{L_p(B_\rho(x_0) \times (s, T))} \leq C\eta^{2-\alpha}\|v\|_{W_p^{2,1}(B_{\rho+\eta}(x_0) \times (s, T))} + C \left(\max_{\mathbb{R}^n \times [s, T]} |v| + \max_{B_{\rho+1}(x_0) \times [s, T]} |\nabla_x v| \right) \cdot \begin{cases} (1 + \eta^{1-\alpha}), & \alpha \neq 1 \\ (1 - \log \eta), & \alpha = 1 \end{cases}.$$

Proof. Let us break the integral into three parts.

$$\begin{aligned} |Iv(x, t)| &= \left| \int_{\mathbb{R}^n} [v(x+y, t) - v(x, t) - y \cdot \nabla_x v(x, t) 1_{\{|y| \leq 1\}}] \nu(dy) \right| \\ &\leq \int_{|y| \leq \eta} \nu(dy) \int_0^1 dz (1-z) \sum_{i,j=1}^n \left| y^i y^j \frac{\partial^2}{\partial x^i \partial x^j} v(x+zy, t) \right| \\ &\quad + \int_{\eta < |y| \leq 1} \nu(dy) |v(x+y, t) - v(x, t) - y \cdot \nabla_x v(x, t)| + \int_{|y| > 1} \nu(dy) |v(x+y, t) - v(x, t)| \\ &\leq \sum_{i,j=1}^n \int_{|y| \leq \eta} |y|^2 \nu(dy) \int_0^1 dz \left| \frac{\partial^2}{\partial x^i \partial x^j} v(x+zy, t) \right| \\ &\quad + \int_{\eta < |y| \leq 1} \nu(dy) |v(x+y, t) - v(x, t) - y \cdot \nabla_x v(x, t)| + \int_{|y| > 1} \nu(dy) |v(x+y, t) - v(x, t)| \\ &\triangleq \sum_{i,j=1}^n I_{i,j}(x, t) + I_2(x, t) + I_3(x, t). \end{aligned}$$

In the following, we will estimate the L_p -norm of each term respectively.

(6.2)

$$\begin{aligned}
& \|I_{ij}(\cdot, t)\|_{L_p(B_\rho)}^p \\
&= \int_{B_\rho} dx \left[\int_{|y| \leq \eta} |y|^2 \nu(dy) \int_0^1 dz |\partial_{x^i x^j}^2 v(x + zy, t)| \right]^p \leq \int_{B_\rho} dx \int_0^1 dz \left[\int_{|y| \leq \eta} \nu(dy) |y|^2 |\partial_{x^i x^j}^2 v(x + zy, t)| \right]^p \\
&\leq M^p \int_{B_\rho} dx \int_0^1 dz \left[\int_{|y| \leq \eta} dy |y|^{2-n-\alpha} |\partial_{x^i x^j}^2 v(x + zy, t)| \right]^p \\
&\leq M^p \int_{B_\rho} dx \int_0^1 dz \left(\int_{|y| \leq \eta} dy |y|^{2-n-\alpha} \right)^{\frac{p}{q}} \cdot \left(\int_{|y| \leq \eta} dy |y|^{2-n-\alpha} |\partial_{x^i x^j}^2 v(x + zy, t)|^p \right) \\
&= M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^{\frac{p}{q}} \cdot \int_0^1 dz \int_{|y| \leq \eta} dy |y|^{2-n-\alpha} \int_{B_\rho} dx |\partial_{x^i x^j}^2 v(x + zy, t)|^p \\
&\leq M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^{\frac{p}{q}} \cdot \int_0^1 dz \int_{|y| \leq \eta} dy |y|^{2-n-\alpha} \|\partial_{x^i x^j}^2 v(\cdot, t)\|_{L_p(B_{\rho+\eta})}^p \\
&= M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^p \cdot \|\partial_{x^i x^j}^2 v(\cdot, t)\|_{L_p(B_{\rho+\eta})}^p.
\end{aligned}$$

Here the first inequality follows from Fubini's Theorem and Jensen's inequality with respect to the Lebesgue measure dz . Assumption (H5) is used in the second inequality. The third inequality follows from Hölder inequality with $1/p + 1/q = 1$. In the second equality, $|S_1(0)|$ is the surface area of the unit ball in \mathbb{R}^n . Note that $x + zy \in B_{\rho+\eta}$ when $x \in B_\rho$, $z \in (0, 1)$ and $|y| \leq \eta$, the fourth inequality follows.

For I_2 and I_3 , noting that $x + y \in B_{\rho+1}$ when $x \in B_\rho$ and $|y| \leq 1$, we have

$$(6.3) \quad \|I_2(\cdot, t)\|_{L_p(B_\rho)} \leq C \cdot \max_{B_{\rho+1} \times [s, T]} |\nabla_x v| \cdot \begin{cases} (1 + \eta^{1-\alpha}), & \alpha \neq 1 \\ (1 - \log \eta), & \alpha = 1 \end{cases} \quad \text{and}$$

$$(6.4) \quad \|I_3(\cdot, t)\|_{L_p(B_\rho)} \leq C \cdot \max_{\mathbb{R}^n \times [s, T]} |v| \cdot \int_{|y| > 1} \nu(dy).$$

Combining (6.2) - (6.4), (6.1) follows from $\|Iv\|_{L_p(B_\rho \times (s, T))} \triangleq \left[\int_s^T \|Iv(\cdot, t)\|_{L_p(B_\rho)}^p dt \right]^{1/p}$ and $\|\partial_{x^i x^j}^2 v\|_{L_p(B_{\rho+\eta} \times (s, T))} \leq \|v\|_{W_p^{2,1}(B_{\rho+\eta} \times (s, T))}$ (see Definition 4.1). \square

In (6.1), when $\alpha \in [0, 1)$ (finite variation jumps), the factors of η in both terms on the right-hand-side converge to 0 as $\eta \rightarrow 0$. Therefore, the L_p -norm of Iv on the domain $B_\rho(x_0) \times (s, T)$ essentially only depends on $\max_{\mathbb{R}^n \times [s, T]} |v|$ and $\max_{B_{\rho+1} \times [s, T]} |\nabla_x v|$. This can be also confirmed by working with the reduced integral form $I^f v$ in (3.2).

On the contrary, when $\alpha \in [1, 2)$ (infinite variation jumps), the factor $1 + \eta^{1-\alpha}$ (or $1 - \log \eta$) in (6.1) will blow up as $\eta \rightarrow 0$ (a similar phenomenon was also observed in Lemma 1.1 of [6] pp.206 for L_p -norm on \mathbb{R}^n). Therefore, it is important to note that the L_p -norm of Iv on the domain $B_\rho(x_0) \times (s, T)$ actually depends on $W_p^{2,1}$ -norm of v on a larger domain $B_{\rho+\eta}(x_0) \times (s, T)$. Because of the expansion of the domain, instead of using the boundary estimate in Theorem 9.1 in [22] pp. 342, we will use the interior estimation technique in Theorem 10.1 in [22] pp. 351 to prove Theorem 5.1 in the following.

Proof of Theorem 5.1. Let us choose a cut-off function $\zeta^\delta(x, t)$ such that

$$\zeta^\delta(x, t) = \begin{cases} 1 & (x, t) \in B_\rho \times (\delta, T) \\ 0 & (x, t) \in \mathbb{R}^n \times (0, T) \setminus B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T) \end{cases}$$

Here the constant $\delta \in (0, s)$ will be determined later. This cut-off function can be chosen such that

$$(6.5) \quad |\partial_{x^i} \zeta^\delta| \leq \frac{C_1}{\delta}, \quad |\partial_{x^i x^j}^2 \zeta^\delta| \leq \frac{C_2}{\delta^2} \quad \text{and} \quad |\partial_t \zeta^\delta| \leq \frac{C_3}{\delta},$$

for $i, j \leq n$ and some constants C_1, C_2 and C_3 . Please see Figure 1 for the domains used in this proof.

Defining $u(x, t) = \zeta^\delta(x, t)v(x, t)$, it satisfies

$$\begin{aligned} (\partial_t - \mathcal{L}_D + r) u(x, t) &= \zeta^\delta \cdot Iv(x, t) + \zeta^\delta \cdot f(x, t) + h(x, t), \quad (x, t) \in B_{\rho+\frac{\delta}{4}} \times (0, T), \\ u(x, t) &= 0, \quad (x, t) \in \partial B_{\rho+\frac{\delta}{4}} \times (0, T), \\ u(x, 0) &= 0, \quad x \in \overline{B_{\rho+\frac{\delta}{4}}}, \end{aligned}$$

in which $h(x, t) \triangleq \partial_t \zeta^\delta \cdot v - \sum_{i,j=1}^n a_{ij} (\partial_{x^i x^j}^2 \zeta^\delta \cdot v + 2 \partial_{x^i} \zeta^\delta \cdot \partial_{x^j} v) - \sum_{i=1}^n b_i \cdot \partial_{x^i} \zeta^\delta \cdot v$. Appealing to Theorem 9.1 in [22] pp.341, we can find a constant C such that

$$(6.6) \quad \begin{aligned} \|u\|_{W_p^{2,1}(B_{\rho+\frac{\delta}{4}} \times (0, T))} &\leq C \left[\|\zeta^\delta \cdot Iv\|_{L_p} + \|\zeta^\delta \cdot f\|_{L_p} + \|\partial_t \zeta^\delta \cdot v\|_{L_p} + \left\| \sum_{i,j=1}^n a_{ij} \partial_{x^i x^j}^2 \zeta^\delta \cdot v \right\|_{L_p} \right. \\ &\quad \left. + \left\| \sum_{i,j=1}^n 2 a_{ij} \partial_{x^i} \zeta^\delta \cdot \partial_{x^j} v \right\|_{L_p} + \left\| \sum_{i=1}^n b_i \cdot \partial_{x^i} \zeta^\delta \cdot v \right\|_{L_p} \right], \end{aligned}$$

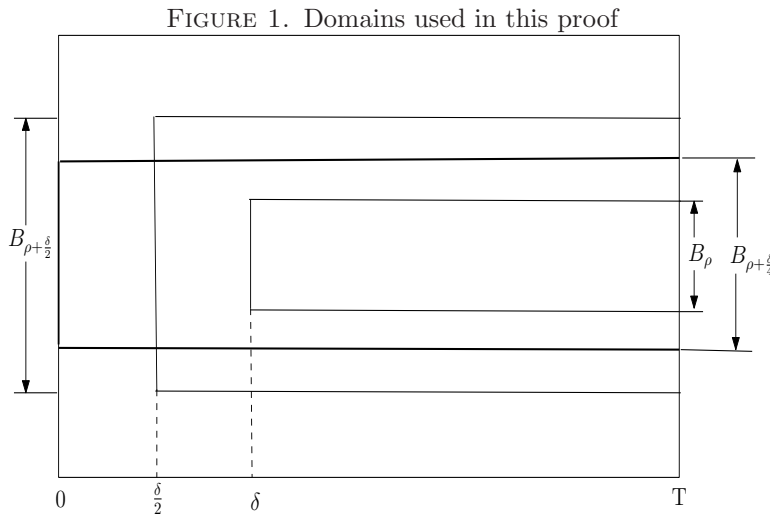
in which all L_p -norms on the right-hand-side are on $B_{\rho+\frac{\delta}{4}} \times (0, T)$.

In the following, we will estimate the terms on the right-hand-side of (6.6) respectively.

(6.7)

$$\begin{aligned} \|\zeta^\delta \cdot Iv\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} &\leq \|Iv\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))} \\ &\leq C \left(\frac{\delta}{4} \right)^{2-\alpha} \|v\|_{W_p^{2,1}(B_{\rho+\frac{\delta}{2}} \times (\frac{\delta}{2}, T))} + C \left(1 + \left(\frac{\delta}{4} \right)^{1-\alpha} \right) \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0, T]} |\nabla_x v| \right]. \end{aligned}$$

Here the first inequality follows from the choice of the cut-off function ζ^δ , the second inequality follows from Lemma 6.1 for $\alpha \neq 1$ case by picking $\eta = \frac{\delta}{4}$ and $s = \frac{\delta}{2}$. When $\alpha = 1$, we also have an estimate similar to (6.7). On



the other hand, we have

$$(6.8) \quad \|\zeta^\delta \cdot f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} \leq \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))}.$$

Moreover, we obtain from (6.5) that

$$(6.9) \quad \begin{aligned} \|\partial_t \zeta^\delta \cdot v\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} &\leq \max_{\mathbb{R}^n \times [0, T]} |v| \cdot \|\partial_t \zeta^\delta\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} \\ &\leq \max_{\mathbb{R}^n \times [0, T]} |v| \left(\int_{B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T) \setminus B_\rho \times (\delta, T)} dt dx \frac{C_3^p}{\delta^p} \right)^{\frac{1}{p}} \\ &\leq C \max_{\mathbb{R}^n \times [0, T]} |v| \cdot \delta^{\frac{1-p}{p}}. \end{aligned}$$

Similarly, thanks to (H7'), we also have

$$(6.10) \quad \left\| \sum_{i,j=1}^n a_{ij} \partial_{x^i x^j}^2 \zeta^\delta \cdot v \right\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} \leq C \max_{\mathbb{R}^n \times [0, T]} |v| \cdot \delta^{\frac{1-2p}{p}},$$

$$(6.11) \quad \left\| \sum_{i,j=1}^n 2 a_{ij} \partial_{x^i} \zeta^\delta \cdot \partial_{x^j} v \right\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} \leq C \max_{B_{\rho+\frac{\delta}{4}} \times [0, T]} |\nabla_x v| \cdot \delta^{\frac{1-p}{p}} \quad \text{and}$$

$$(6.12) \quad \left\| \sum_{i=1}^n b_i \cdot \partial_{x^i} \zeta^\delta \cdot v \right\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0, T))} \leq C \max_{\mathbb{R}^n \times [0, T]} |v| \cdot \delta^{\frac{1-p}{p}}.$$

Plugging (6.7) - (6.12) into (6.6) and noticing the choice of the cut-off function ζ^δ , we obtain

$$(6.13) \quad \begin{aligned} \|v\|_{w_p^{2,1}(B_\rho \times (\delta, T))} &\leq \|u\|_{w_p^{2,1}(B_{\rho+\frac{\delta}{4}} \times (0, T))} \\ &\leq C \left(\frac{\delta}{4} \right)^{2-\alpha} \|v\|_{W_p^{2,1}(B_{\rho+\frac{\delta}{2}} \times (\frac{\delta}{2}, T))} + C \left[1 + \delta^{1-\alpha} + \delta^{\frac{1-p}{p}} + \delta^{\frac{1-2p}{p}} \right] \cdot \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0, T]} |\nabla_x v| \right] \\ &\quad + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))}. \end{aligned}$$

Multiplying δ^2 on both hand side of (6.13) and defining

$$K(\delta) = C \left[\delta^2 + \delta^{3-\alpha} + \delta^{\frac{1+p}{p}} + \delta^{\frac{1}{p}} \right] \cdot \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0, T]} |\nabla_x v| \right] + \delta^2 \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))},$$

we obtain

$$(6.14) \quad \delta^2 \|v\|_{w_p^{2,1}(B_\rho \times (\delta, T))} \leq 4C \left(\frac{\delta}{4} \right)^{2-\alpha} \cdot \left(\frac{\delta}{2} \right)^2 \|v\|_{w_p^{2,1}(B_{\rho+\frac{\delta}{2}} \times (\frac{\delta}{2}, T))} + K(\delta).$$

Let $F(\tau) \triangleq \tau^2 \|v\|_{w_p^{2,1}(B_{\rho+\delta-\tau} \times (\tau, T))}$. The inequality (6.14) gives us the following recursive inequality

$$(6.15) \quad F(\delta) \leq 4C \left(\frac{\delta}{4} \right)^{2-\alpha} F(\delta/2) + K(\delta).$$

Since $\alpha < 2$, we can choose sufficiently small δ such that $4C(\delta/4)^{2-\alpha} \leq \frac{1}{2}$. Therefore, we have from (6.15) that

$$(6.16) \quad F(\delta) \leq \frac{1}{2} F(\delta/2) + K(\delta).$$

On the other hand, thanks to the assumption $v \in W_{p,loc}^{2,1}(\mathbb{R}^n \times (0, T))$, $F(\delta)$ is finite for any $\delta \in (0, \delta_0)$. Iterating the recursive inequality (6.16) gives us

$$F(\delta) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K\left(\frac{\delta}{2^i}\right) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K(\delta) = 2 K(\delta),$$

where the second inequality follows from noticing that $K(\delta)$ is increasing in δ . Therefore, it follows from the definitions of $F(\delta)$ and $K(\delta)$ that

$$\begin{aligned} \|v\|_{W_p^{2,1}(B_\rho \times (s, T))} &\leq \|v\|_{W_p^{2,1}(B_\rho \times (\delta, T))} \\ &\leq 2C \left[1 + \delta^{1-\alpha} + \delta^{\frac{1-p}{p}} + \delta^{\frac{1-2p}{p}} \right] \cdot \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0, T]} |\nabla_x v| \right] + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))} \\ &\leq C_\delta \left[\max_{\mathbb{R}^n \times [0, T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0, T]} |\nabla_x v| + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))} \right]. \end{aligned}$$

□

APPENDIX A. PROOFS OF SEVERAL LEMMAS IN SECTIONS 2, 3 AND 4

Proof of Lemma 2.1. Throughout this proof, in order to distinguish the Euclidean norm in \mathbb{R}^n from the absolute value in \mathbb{R} , we denote the Euclidean norm as $\|\cdot\|$ and the absolute value as $|\cdot|$. Actually, the norm $\|\cdot\|$ is equivalent to the sum of the norms $|\cdot|$ among all components, i.e.,

$$(A-1) \quad \|y\| \leq \sum_{i=1}^n |y^i| \leq n \|y\|, \quad \text{for any } y \in \mathbb{R}^n.$$

Thanks to (A-1), (2.4) and (2.5) can be proved under a slightly weaker assumption (H2) than $\int_{|y|>1} |y|^2 \nu(dy)$, which is the main assumption of Lemma 3.1 in [27]. We will only prove the second and third estimates in (2.5) in what follows.

Following from (1.1) and (2.2), we have for any $\tau \in \mathcal{T}_{0,t}$ that

$$(A-2) \quad \|X_\tau^x - x\| \leq \left\| \int_0^\tau b(X_s^x, s) ds \right\| + \left\| \int_0^\tau \sigma(X_s^x, s) dW_s \right\| + \|\mathcal{J}_\tau^\ell\| + \left\| \lim_{\epsilon \downarrow 0} \mathcal{J}_\tau^\epsilon \right\|.$$

The difference of our proof from the proof of Lemma 3.1 in [27] is the estimation of the large jump term $\|\mathcal{J}_\tau^\ell\|$. We will focus on the estimation of this term in what follows.

First, it follows from (2.2) and the triangle inequality that

$$(A-3) \quad \mathbb{E} \|\mathcal{J}_\tau^\ell\| = \mathbb{E} \left\| \int_0^\tau \int_{\|y\|>1} y \mu(ds, dy) \right\| \leq \mathbb{E} \left\| \int_0^\tau \int_{\|y\|>1} y \tilde{\mu}(ds, dy) \right\| + \mathbb{E} \left\| \int_0^\tau ds \int_{\|y\|>1} y \nu(dy) \right\|.$$

Let us estimate the two terms on the right-hand-side of (A-3) separately. On the one hand, $\int_0^t \int_{\|y\|>1} y \tilde{\mu}(ds, dy)$ is a martingale because of (H2). Hence $\left\| \int_0^t \int_{\|y\|>1} y \tilde{\mu}(ds, dy) \right\|$ is a submartingale (see e.g. Problem 3.7 in [21] pp. 13). It follows from the Optional Sampling Theorem that

$$(A-4) \quad \mathbb{E} \left\| \int_0^\tau \int_{\|y\|>1} y \tilde{\mu}(ds, dy) \right\| \leq \mathbb{E} \left\| \int_0^t \int_{\|y\|>1} y \tilde{\mu}(ds, dy) \right\|.$$

Thanks to (A-1), we can estimate the right-hand-side of (A-4) as follows:

$$\begin{aligned}
\mathbb{E} \left\| \int_0^t \int_{\|y\|>1} y \tilde{\mu}(ds, dy) \right\| &\leq \mathbb{E} \sum_{i=1}^n \left| \int_0^t \int_{\|y\|>1} y^i \tilde{\mu}(ds, dy) \right| \\
&\leq \mathbb{E} \sum_{i=1}^n \left| \int_0^t \int_{\|y\|>1} y^i \mu(ds, dy) \right| + \sum_{i=1}^n \int_0^t ds \int_{\|y\|>1} |y^i| \nu(dy) \\
&\leq \mathbb{E} \int_0^t \int_{\|y\|\geq 1} \sum_{i=1}^n |y^i| \mu(ds, dy) + \int_0^t ds \int_{\|y\|>1} \sum_{i=1}^n |y^i| \nu(dy) \\
&= 2 \int_0^t ds \int_{\|y\|>1} \sum_{i=1}^n |y^i| \nu(dy) \leq 2n \int_{\|y\|>1} \|y\| \nu(dy) \cdot t.
\end{aligned}
\tag{A-5}$$

Here the first and fourth inequalities follow from (A-1). Moreover, the third inequality follows since the Poisson random measure μ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^n$ for each $\omega \in \Omega$. On the other hand, the second term on the right-hand-side of (A-3) can be estimated similarly using (A-1).

Thanks to (A-3) - (A-5), we can find a positive constant C such that $\mathbb{E} \|\mathcal{J}_\tau^\ell\| \leq Ct$ for any $\tau \in \mathcal{T}_{0,t}$. The other three terms on the right-hand-side of (A-2) can be estimated in the same way as in Lemma 3.1 of [27]. In particular, the stochastic integral and the small jump terms are bounded by $Ct^{1/2}$. Moreover, compared to the estimate (3.3) in [27], the boundness of b and σ ensures that the constant C in (2.5) is independent of x .

To prove the third estimate in (2.5), we will still focus on the large jump term. Instead of applying the Doob's inequality as in Lemma 3.1 in [27], we will use properties of μ to derive the following estimate:

$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq s \leq t} \|\mathcal{J}_s^\ell\| \right] &= \mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \int_{\|y\|>1} y \mu(du, dy) \right\| \right] \leq \mathbb{E} \left[\sup_{0 \leq s \leq t} \sum_{i=1}^n \left| \int_0^s \int_{\|y\|>1} y^i \mu(du, dy) \right| \right] \\
&\leq \mathbb{E} \left[\sup_{0 \leq s \leq t} \int_0^s \int_{\|y\|>1} \sum_{i=1}^n |y^i| \mu(du, dy) \right] \leq \mathbb{E} \left[\int_0^t \int_{\|y\|>1} \sum_{i=1}^n |y^i| \mu(du, dy) \right] \\
&= \int_0^t du \int_{\|y\|>1} \sum_{i=1}^n |y^i| \nu(dy) \leq n \int_{\|y\|>1} \|y\| \nu(dy) \cdot t.
\end{aligned}
\tag{A-6}$$

Here the first and fourth inequalities follow from (A-1), the second and the third inequalities hold since μ is a non-negative measure for each $\omega \in \Omega$. The rest proof follows from the same approach used in Lemma 3.1 of [27]. \square

Proof of Lemma 3.1. We shall show the Hölder continuity in x first. Let us break up the integral into two parts:

$$(A-7) \quad |I^f u(x_1, t) - I^f u(x_2, t)| \leq \int_{\mathbb{R}} |u(x_1 + y, t) - u(x_1, t) - u(x_2 + y, t) + u(x_2, t)| \nu(dy) \leq I_1 + I_2, \quad \text{in which}$$

$$(A-8) \quad I_1 = \int_{|y| \leq \epsilon} [|u(x_1 + y, t) - u(x_1, t)| + |u(x_2 + y, t) - u(x_2, t)|] \nu(dy),$$

$$(A-9) \quad I_2 = \int_{|y| > \epsilon} [|u(x_1 + y, t) - u(x_2 + y, t)| + |u(x_1, t) - u(x_2, t)|] \nu(dy).$$

Here the constant $\epsilon \in (0, 1]$ will be determined later. Since $x \rightarrow u(x, t)$ is globally Lipschitz (see Lemma 2.2),

$$|u(x_i + y, t) - u(x_i, t)| \leq L_x |y|, \quad |u(x_1 + y, t) - u(x_2 + y, t)| \leq L_x |x_1 - x_2| \quad \text{and} \quad |u(x_1, t) - u(x_2, t)| \leq L_x |x_1 - x_2|,$$

for $i = 1, 2$. Combining these inequalities with (H5), in which $0 \leq \alpha < 1$, we obtain from (A-8) and (A-9) that

$$(A-10) I_1 \leq \int_{|y| \leq \epsilon} 2L_x |y| \nu(dy) \leq 2L_x M \int_{|y| \leq \epsilon} |y|^{1-n-\alpha} dy = 2L_x M |S_1(0)| \int_0^\epsilon r^{-\alpha} dr = \frac{2L_x M |S_1(0)|}{1-\alpha} \epsilon^{1-\alpha},$$

$$(A-11) I_2 \leq \int_{|y| > \epsilon} 2L_x |x_1 - x_2| \nu(dy) \leq 2L_x |x_1 - x_2| \int_{|y| > 1} \nu(dy) + 2L_x M |x_1 - x_2| \int_{\epsilon < |y| \leq 1} |y|^{-n-\alpha} dy$$

$$= 2L_x |x_1 - x_2| \int_{|y| > 1} \nu(dy) + 2L_x M |S_1(0)| |x_1 - x_2| \cdot \begin{cases} \frac{\epsilon^{-\alpha}-1}{\alpha} & \text{if } 0 < \alpha < 1 \\ -\log \epsilon & \text{if } \alpha = 0, \end{cases}$$

where $|S_1(0)|$ is the surface area of a unit ball in \mathbb{R}^n . Now picking $\epsilon = |x_1 - x_2| \wedge 1$ and noticing that $0 \leq \alpha < 1$, we have

$$(A-12) \quad \epsilon^{1-\alpha} \leq |x_1 - x_2|^{1-\alpha}, \quad \epsilon^{-\alpha} - 1 \leq |x_1 - x_2|^{-\alpha}.$$

Moreover, when $\epsilon = |x_1 - x_2| < 1$,

$$(A-13) \quad -\log \epsilon = \int_{|x_1 - x_2|}^1 \frac{1}{z} dz \leq \int_{|x_1 - x_2|}^1 \frac{1}{z^{1+\beta}} dz = \frac{1}{\beta} (|x_1 - x_2|^{-\beta} - 1) \leq \frac{1}{\beta} |x_1 - x_2|^{-\beta} \quad \forall \beta > 0.$$

Hence choosing $\epsilon = |x_1 - x_2| \wedge 1$, we have $-\log \epsilon \leq \frac{1}{\beta} |x_1 - x_2|^{-\beta}$ for any $\beta > 0$. Combining (A-7) and (A-10) - (A-13), we conclude that

$$\text{when } 0 < \alpha < 1: \quad |I^f u(x_1, t) - I^f u(x_2, t)| \leq \left[\frac{2L_x M |S_1(0)|}{\alpha(1-\alpha)} + 2L_x d^\alpha \int_{|y| > 1} \nu(dy) \right] |x_1 - x_2|^{1-\alpha},$$

$$\text{when } \alpha = 0: \quad |I^f u(x_1, t) - I^f u(x_2, t)| \leq \left[2L_x M |S_1(0)| d^\beta + \frac{2L_x M |S_1(0)|}{\beta} + 2L_x d^\beta \int_{|y| > 1} \nu(dy) \right] |x_1 - x_2|^{1-\beta},$$

in which $\beta \in (0, 1)$ and $d = \max_{x, y \in \Omega} |x - y|$.

Similarly, in order to show the Hölder continuity in t , we also break up the integral term into two parts:

$$(A-14) |I^f u(x, t_1) - I^f u(x, t_2)| \leq \int_{\mathbb{R}} |u(x + y, t_1) - u(x, t_1) - u(x + y, t_2) + u(x, t_2)| \nu(dy) \leq I_1 + I_2, \text{ in which}$$

$$(A-15) \quad I_1 = \int_{|y| \leq \epsilon} [|u(x + y, t_1) - u(x, t_1)| + |u(x + y, t_2) - u(x, t_2)|] \nu(dy),$$

$$(A-16) \quad I_2 = \int_{|y| > \epsilon} [|u(x + y, t_1) - u(x + y, t_2)| + |u(x, t_1) - u(x, t_2)|] \nu(dy).$$

The constant $\epsilon \in (0, 1]$ will be determined later. We can first bound I_1 in (A-15) using (A-10). Then it follows from the semi-Hölder continuity of $t \rightarrow u(x, t)$ (see Lemma 2.2) that

$$(A-17) \quad I_2 \leq \int_{|y| > \epsilon} 2L_t |t_1 - t_2|^{\frac{1}{2}} \nu(dy) = 2L_t |t_1 - t_2|^{\frac{1}{2}} \int_{\epsilon < |y| \leq 1} \nu(dy) + 2L_t |t_1 - t_2|^{\frac{1}{2}} \int_{|y| > 1} \nu(dy)$$

$$\leq 2L_t |t_1 - t_2|^{\frac{1}{2}} \int_{|y| > 1} \nu(dy) + 2L_t M |S_1(0)| |t_1 - t_2|^{\frac{1}{2}} \cdot \begin{cases} \frac{\epsilon^{-\alpha}-1}{\alpha}, & \text{if } 0 < \alpha < 1 \\ -\log \epsilon, & \text{if } \alpha = 0, \end{cases},$$

in which the second inequality follows from (H5) with $0 \leq \alpha < 1$.

Now picking $\epsilon = |t_1 - t_2|^{\frac{1}{2}} \wedge 1$, we have $\epsilon^{1-\alpha} \leq |t_1 - t_2|^{\frac{1-\alpha}{2}}$ and $\epsilon^{-\alpha} - 1 \leq |t_1 - t_2|^{-\frac{\alpha}{2}}$. A calculation in (A-13) gives us that $-\log \epsilon \leq 2|t_1 - t_2|^{-\beta/2}/\beta$ for any $\beta > 0$. Therefore (3.5) and (3.6) follow from combining (A-14), (A-10) and (A-17). \square

Proof of Lemma 3.2. Let us first consider the nonlocal boundary value problem:

$$(A-18) \quad \begin{aligned} (-\partial_t - \mathcal{L} + r) v(x, t) &= 0, \quad (x, t) \in B \times [t_1, t_2], \\ v(x, t) &= u(x, t), \quad (x, t) \in \mathbb{R}^n \times [t_1, t_2] \setminus B \times [t_1, t_2], \end{aligned}$$

in which B is the same domain as in (3.7). The viscosity solution of (A-18) is defined as follows. (See e.g. Definition 12.1 in [9].)

Definition A-1. Any $v \in C^0(\overline{B} \times [t_1, t_2])$ is a viscosity subsolution of (A-18) if

$$(A-19) \quad (-\partial_t - \mathcal{L} + r)\phi(x, t) \leq 0, \quad \text{for } (x, t) \in B \times [t_1, t_2],$$

$$(A-20) \quad \min \{(-\partial_t - \mathcal{L} + r)\phi(x, t), v(x, t) - u(x, t)\} \leq 0, \quad \text{for } (x, t) \in \partial B \times [t_1, t_2] \cup \overline{B} \times t_2,$$

$$(A-21) \quad v(x, t) \leq u(x, t), \quad \text{for } (x, t) \in \mathbb{R}^n \times [t_1, t_2] \setminus \overline{B} \times [t_1, t_2],$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2]) \cap C_1(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x, t) = v(x, t)$ and $\phi(\tilde{x}, \tilde{t}) \geq v(\tilde{x}, \tilde{t})$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [t_1, t_2]$. The viscosity supersolution is also defined analogously. As usual, a viscosity solution is both a subsolution and a supersolution.

Using this definition, it is easy to check that u is a viscosity solution of (A-18). Now the statement of the lemma follows from the equivalence of Definition A-1 and Definition 3.1. The proof this equivalence is only a slight modification of the arguments in the proof of Lemma 2.1 of [30]. \square

Proof of Lemma 4.1. For the notational simplicity, the constant C denotes a generic constant in different places in this proof.

1. Let us first estimate $\max_{\overline{Q_s}} |I\phi|$. Following (2.10), for $(x, t) \in \overline{Q_s}$, we have

$$\begin{aligned} (A-22) |I\phi(x, t)| &\leq \int_{|y| \leq 1} \left| \phi(x + y, t) - \phi(x, t) - \sum_{i=1}^n y^i \partial_{x^i} \phi(x, t) \right| \nu(dy) + \int_{|y| > 1} |\phi(x + y, t) - \phi(x, t)| \nu(dy) \\ &\leq \int_{|y| \leq 1} \sum_{i=1}^n |y^i \partial_{x^i} \phi(z_i, t) - y^i \partial_{x^i} \phi(x, t)| \nu(dy) + 2 \max_{\mathbb{R}^n \times [0, s]} |\phi| \int_{|y| > 1} \nu(dy) \\ &\leq \|\phi\|_{\overline{Q_s^1}}^{(\beta)} \int_{|y| \leq 1} |y|^\beta \nu(dy) + 2 \max_{\mathbb{R}^n \times [0, s]} |\phi| \int_{|y| > 1} \nu(dy) \\ &\leq C \left(\max_{\mathbb{R}^n \times [0, s]} |\phi| + \|\phi\|_{\overline{Q_s^1}}^{(\beta)} \right). \end{aligned}$$

In the second inequality of (A-22), z_i are some vectors in \mathbb{R}^n with $|z_i - x| < |y|$. Therefore, when $x \in \Omega$, we have $x + z_i \in \Omega^1$. The third inequality follows from the Hölder continuity of $\partial_{x^i} \phi$ on $\overline{Q_s^1}$, i.e., $\sum_{i=1}^n |\partial_{x^i} \phi(z_i, t) - \partial_{x^i} \phi(x, t)| \leq \|\phi\|_{\overline{Q_s^1}}^{(\beta)} |y|^{\beta-1}$. We apply (H5) to obtain the last inequality. Note that $\beta > \alpha$, hence $\int_{|y| \leq 1} |y|^{-n+\beta-\alpha} dy$ is integrable.

The proof of the Hölder continuity of $x \rightarrow I\phi(x, t)$ and $t \rightarrow I\phi(x, t)$ are similar to the proof in Lemmas 3.1. Let us check the Hölder continuity in x first. For any $x_1, x_2 \in \Omega$ and $t \in [0, s]$, breaking up the integral term into three parts, we obtain

$$(A-23) |I\phi(x_1, t) - I\phi(x_2, t)| \leq I_1 + I_2 + I_3, \quad \text{in which}$$

$$\begin{aligned} I_1(x, t) &= \int_{|y| \leq \epsilon} [|\phi(x_1 + y, t) - \phi(x_1, t) - y \cdot \nabla_x \phi(x_1, t)| + |\phi(x_2 + y, t) - \phi(x_2, t) - y \cdot \nabla_x \phi(x_2, t)|] \nu(dy), \\ I_2(x, t) &= \int_{\epsilon < |y| \leq 1} [|\phi(x_1 + y, t) - \phi(x_2 + y, t)| + |\phi(x_1, t) - \phi(x_2, t)| + |y| |\nabla_x \phi(x_1, t) - \nabla_x \phi(x_2, t)|] \nu(dy), \\ I_3(x, t) &= \int_{|y| > 1} [|\phi(x_1 + y, t) - \phi(x_2 + y, t)| + |\phi(x_1, t) - \phi(x_2, t)|] \nu(dy). \end{aligned}$$

Here the constant $\epsilon \leq 1$ will be determined later. Let us estimate each integral term separately. An estimate similar to (A-22) shows that

$$(A-24) \quad I_1 \leq 2\|\phi\|_{Q_s^1}^{(\beta)} \int_{|y| \leq \epsilon} |y|^\beta \nu(dy) \leq 2M\|\phi\|_{Q_s^1}^{(\beta)} \int_{|y| \leq \epsilon} |y|^{-n+\beta-\alpha} dy = C\|\phi\|_{Q_s^1}^{(\beta)} \epsilon^{\beta-\alpha}.$$

Thanks to the Lipschitz continuity of $x \rightarrow \phi(x, t)$ and the Hölder continuity of $x \rightarrow \partial_{x_i} \phi(x, t)$, we can estimate I_2 and I_3 as

$$(A-25) I_2 \leq \int_{\epsilon < |y| \leq 1} \left[2 \max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| |x_1 - x_2| + \|\phi\|_{Q_s^1}^{(\beta)} |y| |x_1 - x_2|^{\beta-1} \right] \nu(dy) \\ \leq M \int_{\epsilon < |y| \leq 1} \left[2 \max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| |x_1 - x_2| + \|\phi\|_{Q_s^1}^{(\beta)} |y| |x_1 - x_2|^{\beta-1} \right] |y|^{-n-\alpha} dy \\ = C \max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| |x_1 - x_2| (\epsilon^{-\alpha} - 1) + C \|\phi\|_{Q_s^1}^{(\beta)} |x_1 - x_2|^{\beta-1} \cdot \begin{cases} \epsilon^{1-\alpha} - 1 & \text{when } 1 < \alpha < 2, \\ -\log \epsilon & \text{when } \alpha = 1. \end{cases},$$

$$(A-26) I_3 \leq 2 \max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| |x_1 - x_2| \int_{|y| > 1} \nu(dy).$$

Now pick $\epsilon = |x_1 - x_2|^{1/2} \wedge 1$. Note that $1 \leq \alpha < 2$, we obtain $\epsilon^{\beta-\alpha} \leq |x_1 - x_2|^{\frac{\beta-\alpha}{2}}$, $\epsilon^{-\alpha} - 1 \leq |x_1 - x_2|^{-\frac{\alpha}{2}}$, $\epsilon^{1-\alpha} - 1 \leq |x_1 - x_2|^{\frac{1-\alpha}{2}}$ and $-\log \epsilon \leq \frac{1}{\delta} |x_1 - x_2|^{-\delta}$ for any $\delta > 0$ (see (A-13)). Since $\beta > 1$, we will choose $\delta = \frac{\beta-1}{2}$ in the following. Concluding from these inequalities and (A-23) - (A-26), we obtain

$$(A-27) \quad |I\phi(x_1, t) - I\phi(x_2, t)| \leq C_\Omega \left(\max_{\mathbb{R}^n \times [0, s]} |\nabla_x \phi| + \|\phi\|_{Q_s^1}^{(\beta)} \right) |x_1 - x_2|^{\frac{\beta-\alpha}{2}},$$

where C_Ω is a sufficiently large constant independent of x_1, x_2 and t .

For the Hölder continuity of $t \rightarrow I\phi(x, t)$, since $\phi \in H^{\beta, \frac{\beta}{2}}(\overline{Q_s^1})$, it follows from Definition 4.1 that

$$\sum_{i=1}^n |\partial_{x_i} \phi(x, t_1) - \partial_{x_i} \phi(x, t_2)| \leq \|\phi\|_{Q_s^1}^{(\beta)} |t_1 - t_2|^{\frac{\beta-1}{2}}, \quad \text{for } x \in \Omega \text{ and } t_1, t_2 \in [0, s].$$

Picking $\epsilon = |x_1 - x_2|^{\frac{1}{4}} \wedge 1$, an estimation similar to Lemma 3.1 gives us

$$(A-28) \quad |I\phi(x, t_1) - I\phi(x, t_2)| \leq C_\Omega \left(\tilde{L}_t + \|\phi\|_{Q_s^1}^{(\beta)} \right) |t_1 - t_2|^{\frac{\beta-\alpha}{4}},$$

where C_Ω is a sufficiently large constant independent of x, t_1 and t_2 .

Now the first part of the lemma follows from (A-22), (A-27) and (A-28).

2. Noting that $\max_{D_s} |\phi| \leq \|\phi\|_{D_s}^{(\beta)}$ and $\max_{t_1, t_2 \in [0, s]} \frac{|\phi(x, t_1) - \phi(x, t_2)|}{|t_1 - t_2|^{\frac{1}{2}}} \leq s^{\frac{\beta-1}{2}} \|\phi\|_{D_s}^{(\beta)}$ (see Definition 4.1), the second part of the lemma follows from the same argument which we used in the first part of the proof. \square

Proof of Lemma 5.3. For any $R_0 > 0$, let us consider the following function

$$w(x, t) = \frac{m}{f(R_0)} [f(|x|) + C_1 t] + v(x, t),$$

where $f(R) = \frac{R^2}{1+R}$ and the positive constant C_1 will be determined later. It is clear that $f(R)$ is an increasing function on $(0, +\infty)$ and $\lim_{R \rightarrow +\infty} f(R) = +\infty$. On the other hand, $|\partial_{x^i} f(|x|)| \leq \frac{|x|(2+|x|)}{(1+|x|)^2} < 1$ for any $i \leq n$. Moreover, one can also check that $\lim_{|x| \rightarrow +\infty} |\partial_{x^i x^j}^2 f(|x|)| = 0$ and $\lim_{|x| \rightarrow 0} |\partial_{x^i x^j}^2 f(|x|)| = 2\delta_{ij}$ for any $i, j \leq n$. Therefore both $\partial_{x^i} f(|x|)$ and $\partial_{x^i x^j}^2 f(|x|)$ are bounded on \mathbb{R}^n . Thanks to these properties, we can find an upper

bound for $|If(|x|)|$ as follows:

$$\begin{aligned}
(A-29) \quad |If(|x|)| &= \left| \int_{\mathbb{R}^n} \left[f(|x+y|) - f(|x|) - \sum_{i=1}^n y^i \partial_{x^i} f(|x|) 1_{\{|y| \leq 1\}} \right] \nu(dy) \right| \\
&\leq \int_{|y| \leq 1} \nu(dy) \int_0^1 dz (1-z) \sum_{i,j=1}^n |y^i y^j| |\partial_{x^i x^j}^2 f(|x+zy|)| + \int_{|y| > 1} \nu(dy) |f(|x+y|) - f(|x|)| \\
&\leq C \left(\int_{|y| \leq 1} |y|^2 \nu(dy) + \int_{|y| > 1} |y| \nu(dy) \right) < +\infty,
\end{aligned}$$

for some sufficiently large constant $C > 0$. Here the last inequality in (A-29) follows from (2.3) and (H2).

Now, applying the parabolic integro-differential operator to w , we obtain

$$\begin{aligned}
(\partial_t - \mathcal{L}_D - I + c) w(x, t) &\geq (\partial_t - \mathcal{L}_D - I + c) \left[\frac{m}{f(R_0)} (f(|x|) + C_1 t) \right] \\
&= \frac{m}{f(R_0)} \left[C_1 - \sum_{i,j=1}^n a_{ij} \partial_{x^i x^j}^2 f(|x|) - \sum_{i=1}^n b_i \partial_{x^i} f(|x|) + c f(|x|) - If(|x|) \right],
\end{aligned}$$

where the first inequality follows from the assumption that $(\partial_t - \mathcal{L}_D - I + c) v(x, t) \geq 0$. We can choose a sufficiently large constant C_1 independent of R_0 such that

$$(A-30) \quad (\partial_t - \mathcal{L}_D - I + c) w(x, t) > 0, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, T].$$

This is because $\partial_{x^i x^j}^2 f(|x|)$, $\partial_{x^i} f(|x|)$ and coefficients a_{ij} , b_i , c are all bounded, $c \geq 0$ and $|If(|x|)|$ is bounded.

On the other hand, $w(x, 0) = \frac{m}{f(R_0)} f(|x|) + v(x, 0) \geq 0$ thanks to the assumption $v(x, 0) \geq 0$. Moreover, when $|x| = R_0$, $w(x, t) = \frac{m}{f(R_0)} (f(R_0) + C_1 t) + v(x, t) \geq m + v(x, t) \geq 0$ due to the assumption $v(x, t) \geq -m$. Furthermore, when $|x| > R_0$, we also have $w(x, t) \geq m + v(x, t) \geq 0$ since $f(R)$ is an increasing function. Therefore, we claim that $w(x, t) \geq 0$ for $(x, t) \in B_{R_0} \times (0, T_0]$. Indeed, if there are some points $(x, t) \in B_{R_0} \times (0, T_0]$ such that $w(x, t) < 0$, $w(x, t)$ must take its negative minimum at some point $(x_0, t_0) \in B_{R_0} \times (0, T_0]$. Since $w(x, t) \geq 0$ for $|x| \geq R_0$, we have $w(x_0, t_0) \leq w(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, T]$. As a result, we obtain $\partial_t w(x_0, t_0) \leq 0$, $\sum_{i=1}^n b_i \partial_{x^i} w(x_0, t_0) = 0$ and $\sum_{i,j=1}^n a_{ij} \partial_{x^i x^j}^2 w(x_0, t_0) \geq 0$ (see e.g. Lemma 1 in [14] pp. 34). Moreover, $Iw(x_0, t_0) \geq 0$, since w achieves its minimum at (x_0, t_0) and $\nabla_x w(x_0, t_0) = 0$. Therefore, we have

$$(\partial_t - \mathcal{L}_D - I + r) w(x_0, t_0) \leq 0,$$

which contradicts (A-30).

Now, for any point $(x, t) \in \mathbb{R}^n \times (0, T]$, taking $R_0 \rightarrow +\infty$, we have $v(x, t) \geq 0$ since $\lim_{R_0 \rightarrow +\infty} f(R_0) = +\infty$. \square

Proof of Lemma 5.4. First, thanks to Lemma 5.2, $|v^\epsilon|$ is bounded on $\mathbb{R}^n \times [0, T]$. In the following, we will show it is bounded uniformly in ϵ . It follows from (5.3) (i) that $(\partial_t - \mathcal{L}_D - I + r) v^\epsilon = -p_\epsilon(v^\epsilon - g^\epsilon) \geq 0$. Note that $v^\epsilon(x, 0) = g^\epsilon(x) \geq 0$ (see (4.5)), the first inequality in the statement follows from Lemma 5.3 directly. On the other hand, defining $u = K + 1 - v^\epsilon$, u satisfies

$$(A-31) \quad (\partial_t - \mathcal{L}_D - I + r) u = r(K + 1) + p_\epsilon(v^\epsilon - g^\epsilon), \quad (x, t) \in \mathbb{R}^n \times (0, T].$$

It follows from (4.5) and (5.3) (ii) that $p_\epsilon(K + 1 - g^\epsilon) = 0$ with $\epsilon \leq \epsilon_0 \leq 1$. Combining with (A-31) and the mean value theorem, we obtain

$$(A-32) \quad (\partial_t - \mathcal{L}_D - I + r) u + p_\epsilon(K + 1 - g^\epsilon) - p_\epsilon(v^\epsilon - g^\epsilon) = \left[\partial_t - \mathcal{L}_D - I + r + p'_\epsilon(y) \right] u = r(K + 1) \geq 0,$$

for some $y \in \mathbb{R}$. Note that both $K + 1 - g^\epsilon$ and $v^\epsilon - g^\epsilon$ are bounded, p'_ϵ is bounded in any bounded domain. Therefore, we have that $r + p'_\epsilon(y)$ is bounded and nonnegative (see (5.3) (iv)). Applying Lemma 5.3 to u and picking $c = r + p'_\epsilon(y)$, we obtain $u(x, t) = K + 1 - v^\epsilon(x, t) \geq 0$ on $\mathbb{R}^n \times [0, T]$. \square

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